

**Errata and Supplements to
Discrete Convex Analysis (SIAM, 2003)
1st printing (hard cover) (ISBN 0-89871-540-7)**

- Page 77, line 2 from bottom: $+\infty \leq +\infty \implies +\infty \geq +\infty$
- Page 94, line 4 from bottom: Delete “is as follows.”
- Page 109, line 12: $4 \min(\lambda_j, \lambda_k) (\geq 4/N) \implies 2 \min(\lambda_j, \lambda_k) (\geq 2/N)$
- Page 113, line 12 from bottom (last paragraph of the proof of Theorem 4.18):
Let p be such that the optimal solutions to (A) with respect to p form a minimal face of the feasible region of (A).
- Page 117, line 1 (below Figure 4.1):
 $Q \subseteq \mathbf{Z}^V$ is an M^\sharp -convex set $\implies Q \subseteq \mathbf{Z}^V$ is said to be an M^\sharp -convex set
- Page 133, (6.2): $\Delta f(z, v, u) \implies \Delta f(z; v, u)$
- Page 139, Proposition 6.8 (2): Condition (6.26) follows from (6.27), and hence (6.26) is redundant.
- Page 143, three lines above Theorem 6.13:
 $f_{[a,b]}$ for an integer interval $[a, b] \implies$
 $f_{[a,b]}$ for the restriction of f to an integer interval $[a, b]$
- Page 143, Theorem 6.13 (8) [Convolution of M-convex functions]
The proof here makes use of transformation by a network, but an alternative direct proof can be found in:
K. Murota: On infimal convolution of M-convex functions, RIMS Kokyuroku, No.1371 (2004), 20–26.
- Page 151, Theorem 6.30, Proof of Claim 1:
The final step reads: “This shows (B-EXC₊[\mathbf{R}]) for \bar{B} . Therefore, B is an M-convex set.” Before we can argue in this way, we have to verify $\bar{B} \cap \mathbf{Z}^V = B$, which is possible.
- Page 152, Section 6.8: A characterization of gross substitutes property in terms of an exchange property is also found in:
H. Reijnierse, A. van Gallekom, and J. A. M. Potters: Verifying gross substitutability, *Economic Theory*, **20** (2002), 767–776.

- Page 172, Proof of Theorem 6.74:
 “Theorem 6.4 can be strengthened to a statement that $(\text{M-EXC}[\mathbf{Z}])$ and $(\text{M-EXC}_{\text{loc}}[\mathbf{Z}])$ are equivalent if $\text{dom } f$ satisfies (Q-EXC_w) . (This can be shown by modifying the proof of Claim 2 in the proof of Theorem 6.4.)”
 The detail of the argument can be found in a memorandum of A. Shioura: Level set characterization of M-convex functions (February 1998); see Claim 2 on page 6.
- Page 185, Theorem 7.14 [L-optimality criterion]
 The proof here makes use of the optimality criterion for integrally convex functions, but an alternative direct proof can be found in:
 K: Murota: A proof of the L-optimality criterion theorem, unpublished note, July 2004,
<https://kzmurota.fpark.tmu.ac.jp/paper/loptimality04.pdf>
- Page 186, Theorem 7.17
 “with a bounded nonempty effective domain”
 \implies “that is convex-extensible or has a bounded nonempty effective domain”
 For an L-convex function g , the boundedness assumption on the effective domain is intended to mean the boundedness of $\text{dom } g$ intersected with a coordinate plane $\{p \mid p(v) = 0\}$ for some $v \in V$. Note that the effective domain of an L-convex function has the invariance in the direction of $\mathbf{1}$.
- Page 219, Theorem 8.17 [M-convex intersection theorem]
 The proof here makes use of the M-separation theorem, but an alternative direct proof can be found in:
 K. Murota: A proof of the M-convex intersection theorem, RIMS Kokyuroku, No.1371 (2004), 13–19.
- Page 228, line 5, Theorem 8.33: $f_2(x + \chi_{u_{i+1}} - \chi_{v_i}) \implies f_2(x - \chi_{u_{i+1}} + \chi_{v_i})$
- Page 228, line 13, Theorem 8.34:
 $f_2(x^\alpha + \alpha(\chi_{u_{i+1}} - \chi_{v_i})) \implies f_2(x^\alpha - \alpha(\chi_{u_{i+1}} - \chi_{v_i}))$
- Page 231, Proof of Theorem 8.42:
 There was a confusion between L_2 -convexity and L_2^{\natural} -convexity. Please change the first paragraph of the proof (“It suffices to consider found in Murota–Shioura [153].”) to the following text:

To emphasize the essence we give a proof for the integral convexity of an L_2 -convex set, from which the integral convexity of an L_2^{\natural} -convex set follows. This implies, by Theorem 3.29, the integral convexity of an L_2^{\natural} -convex function g with a bounded effective domain, since $\arg \min g[-x]$ is an L_2^{\natural} -convex set for any $x \in \mathbf{R}^V$ by Proposition 8.40. A complete proof can be found in Murota–Shioura [153].

Integral convexity of an L_2^{\natural} -convex set (L_2 -convex set) also follows from its box-total dual integrality (see Theorem 5.1 of S. Moriguchi and K. Murota: Note on the polyhedral description of the Minkowski sum of two L-convex sets, *Japan Journal of Industrial and Applied Mathematics*, Vol.40 (2023), No.1, 223–263. <https://doi.org/10.1007/s13160-022-00512-3> (Open access))

- Page 305, Section 10.3.1:

A detailed analysis of the steepest descent algorithm for L-convex functions can be found in:

K. Murota and A. Shioura: Exact bounds for steepest descent algorithms of L-convex function minimization, *Operations Research Letters*, **42** (2014), 361–366.

- Page 331, $(-M^{\natural}\text{-SWGS}[\mathbf{Z}])$: For $x \in \arg \min U[-p] \implies$ For $x \in \arg \max U[-p]$
- Page 333, “(SNC) for $\tilde{U} \implies M^{\natural}$ -concavity for U ”:

The converse of this statement is also true, that is,

$$(\text{SNC}) \text{ for } \tilde{U} \iff M^{\natural}\text{-concavity for } U$$

holds. See

K. Murota: Multiple exchange property for M^{\natural} -concave functions and valuated matroids, *Mathematics of Operations Research*, **43** (2018), 781–788.

<https://doi.org/10.1287/moor.2017.0882> , arXiv: <http://arxiv.org/abs/1608.07021>

K. Murota: A stronger multiple exchange property for M^{\natural} -concave functions, *Japan Journal of Industrial and Applied Mathematics*, **35** (2018), 411–421.

<https://doi.org/10.1007/s13160-017-0278-4> , arXiv: <http://arxiv.org/abs/1706.09222>

- Page 365, [39]: P. G. Doyle and J. L. Snell: *Random Walks and Electrical Networks*, Mathematical Association of America, Washington DC, 1984.
- Page 369, [105] (S. Iwata and M. Shigeno): (2003) \implies (2002)
- Page 372, [143] (K. Murota): (1999) \implies (1998)
- Page 373, [153] (K. Murota and A. Shioura)
and by Favati–Tardella \implies and Favati–Tardella
- Page 375, [192] (A. Shioura): (2003) \implies (2004)
- Page 376, [202]: D. M. Topkis, \implies D. M. Topkis:

Updates of bib-infor:

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- [156] should be changed to:
K. Murota and A. Shioura: Fundamental properties of M-convex and L-convex functions in continuous variables, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, **E87-A** (2004), 1042–1052.
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