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Errata and Supplements to
Discrete Convex Analysis (SIAM, 2003)
1st printing (hard cover) (ISBN 0-89871-540-7)

- Page 77, line 2 from bottom: $+\infty \leq +\infty \quad \Longrightarrow \quad +\infty \geq +\infty$
- Page 94, line 4 from bottom: Delete "is as follows."
- Page 109, line 12: $4\min(\lambda_j, \lambda_k)$ $(\geq 4/N)$ $\Longrightarrow$ $2\min(\lambda_j, \lambda_k)$ $(\geq 2/N)$
- Page 113, line 12 from bottom (last paragraph of the proof of Theorem 4.18):
  Let p be such that the optimal solutions to (A) with respect to p form a minimal face of the feasible region of (A).
- Page 117, line 1 (below Figure 4.1):
  $Q \subseteq \mathbf{Z}^V$ is an $M^\natural$-*convex set* $\Longrightarrow$ $Q \subseteq \mathbf{Z}^V$ is said to be an $M^\natural$-*convex set*
- Page 133, (6.2): $\Delta f(z, v, u) \quad \Longrightarrow \quad \Delta f(z; v, u)$
- Page 139, Proposition 6.8 (2): Condition (6.26) follows from (6.27), and hence (6.26) is redundant.
- Page 143, three lines above Theorem 6.13:

  $f_{[a,b]}$ for an integer interval $[a, b]$ $\Longrightarrow$
  $f_{[a,b]}$ for the restriction of f to an integer interval $[a, b]$
- Page 143, Theorem 6.13 (8) [Convolution of M-convex functions]
  The proof here makes use of transformation by a network, but an alternative direct proof can be found in:
  K. Murota: On infimal convolution of M-convex functions, RIMS Kokyuroku, No.1371 (2004), 20–26.
- Page 151, Theorem 6.30, Proof of Claim 1:

  The final step reads: "This shows (B-EXC$_+$[$\mathbf{R}$]) for $\overline{B}$. Therefore, B is an M-convex set." Before we can argue in this way, we have to verify $\overline{B} \cap \mathbf{Z}^V = B$, which is possible.
- Page 152, Section 6.8: A characterization of gross substitutes property in terms of an exchange property is also found in:
  H. Reijnierse, A. van Gallekom, and J. A. M. Potters: Verifying gross substitutability, *Economic Theory*, **20** (2002), 767–776.

- Page 172, Proof of Theorem 6.74:

  "Theorem 6.4 can be strengthened to a statement that (M-EXC[$\mathbf{Z}$]) and (M-EXC$_{\rm loc}$[$\mathbf{Z}$]) are equivalent if $\mathrm{dom}\, f$ satisfies (Q-EXC$_{\rm w}$). (This can be shown by modifying the proof of Claim 2 in the proof of Theorem 6.4.)"

  The detail of the argument can be found in a memorandum of A. Shioura: Level set characterization of M-convex functions (February 1998); see Claim 2 on page 6.

- Page 185, Theorem 7.14 [L-optimality criterion]
  The proof here makes use of the optimality criterion for integrally convex functions, but an alternative direct proof can be found in:
  K: Murota: A proof of the L-optimality criterion theorem, unpublished note, July 2004,
  https://kzmurota.fpark.tmu.ac.jp/paper/loptimality04.pdf

- Page 186, Theorem 7.17
  "with a bounded nonempty effective domain"
  $\Longrightarrow$ "that is convex-extensible or has a bounded nonempty effective domain"
  For an L-convex function g, the boundedness assumption on the effective domain is intended to mean the boundedness of $\mathrm{dom}\, g$ intersected with a coordinate plane $\{p \mid p(v) = 0\}$ for some $v \in V$. Note that the effective domain of an L-convex function has the invariance in the direction of $\mathbf{1}$.

- Page 219, Theorem 8.17 [M-convex intersection theorem]
  The proof here makes use of the M-separation theorem, but an alternative direct proof can be found in:
  K. Murota: A proof of the M-convex intersection theorem, RIMS Kokyuroku, No.1371 (2004), 13–19.

- Page 228, line 5, Theorem 8.33: $f_2(x + \chi_{u_{i+1}} - \chi_{v_i}) \quad \Longrightarrow \quad f_2(x - \chi_{u_{i+1}} + \chi_{v_i})$

- Page 228, line 13, Theorem 8.34:
  $f_2(x^\alpha + \alpha(\chi_{u_{i+1}} - \chi_{v_i})) \quad \Longrightarrow \quad f_2(x^\alpha - \alpha(\chi_{u_{i+1}} - \chi_{v_i}))$

- Page 231, Proof of Theorem 8.42:
  There was a confusion between L$_2$-convexity and L$_2^\natural$-convexity. Please change the first paragraph of the proof ("It suffices to consider ..... found in Murota–Shioura [153].") to the following text:

  > To emphasize the essence we give a proof for the integral convexity of an L$_2$-convex set, from which the integral convexity of an L$_2^\natural$-convex set follows. This implies, by Theorem 3.29, the integral convexity of an L$_2^\natural$-convex function g with a bounded effective domain, since $\arg\min g[-x]$ is an L$_2^\natural$-convex set for any $x \in \mathbf{R}^V$ by Proposition 8.40. A complete proof can be found in Murota–Shioura [153].

Integral convexity of an $L_2^\natural$-convex set (L_2-convex set) also follows from its box-total dual integrality (see Theorem 5.1 of S. Moriguchi and K. Murota: Note on the polyhedral description of the Minkowski sum of two L-convex sets, Japan Journal of Industrial and Applied Mathematics, Vol.40 (2023), No.1, 223–263. https://doi.org/10.1007/s13160-022-00512-3 (Open access))

- Page 305, Section 10.3.1:

  A detailed analysis of the steepest descent algorithm for L-convex functions can be found in:

  K. Murota and A. Shioura: Exact bounds for steepest descent algorithms of L-convex function minimization, *Operations Research Letters*, **42** (2014), 361–366.

- Page 331, ($-\mathrm{M}^\natural$-SWGS[$\mathbf{Z}$]): For $x \in \arg\min U[-p]$ $\Longrightarrow$ For $x \in \arg\max U[-p]$

- Page 333, "(SNC) for $\tilde{U} \Longrightarrow \mathrm{M}^\natural$-concavity for U":

  The converse of this statement is also true, that is,

  $$\text{(SNC) for } \tilde{U} \iff \mathrm{M}^\natural\text{-concavity for } U$$

  holds. See K. Murota: Multiple exchange property for $\mathrm{M}^\natural$-concave functions and valuated matroids, Mathematics of Operations Research, **43** (2018), 781-788. https://doi.org/10.1287/moor.2017.0882 , arXiv: http://arxiv.org/abs/1608.07021

- Page 365, [39]: P. G. Doyle and J. L. Snell: *Random Walks and Electrical Networks*, Mathematical Association of America, Washington DC, 1984.

- Page 369, [105] (S. Iwata and M. Shigeno): (2003) $\Longrightarrow$ (2002)

- Page 372, [143] (K. Murota): (1999) $\Longrightarrow$ (1998)

- Page 375, [192] (A. Shioura): (2003) $\Longrightarrow$ (2004)

- Page 376, [202]: D. M. Topkis, $\Longrightarrow$ D. M. Topkis:

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- [156] should be changed to:
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