

“Matrices and Matroids for Systems Analysis”

Theorem 5.1.8 (page 275) の証明の詳細

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5.1.3 Matrix Pencil and Kronecker Form

A polynomial matrix $A(s) = (A_{ij}(s))$ with $\deg_s A_{ij}(s) \leq 1$ for all (i, j) is called a *pencil*. Obviously, a pencil $A(s)$ can be represented as $A(s) = sX + Y$ in terms of a pair of constant matrices X and Y . A pencil $A(s)$ is said to be *regular* if it is square and $\det A(s)$ is a nonvanishing polynomial. A pencil is called *singular* if it is not regular.

A pencil can be brought into a canonical block-diagonal matrix by means of *strict equivalence* $PA(s)Q$ using constant nonsingular matrices P and Q . The block-diagonal matrix is known as the Kronecker form (Gantmacher [87, Chap. XII]). For $m \geq 1$ and $\varepsilon \geq 0$, we define an $m \times m$ bidiagonal matrix $N_m(s)$ and an $\varepsilon \times (\varepsilon + 1)$ bidiagonal matrix $L_\varepsilon(s)$ by

$$N_m(s) = \begin{pmatrix} 1 & s & & & \\ & 1 & s & & \\ & & \ddots & \ddots & \\ & & & \ddots & s \\ & & & & 1 \end{pmatrix}, \quad L_\varepsilon(s) = \begin{pmatrix} 1 & s & & & \\ & 1 & s & & \\ & & \ddots & \ddots & \\ & & & \ddots & s \\ & & & & 1 & s \end{pmatrix}.$$

For $\eta \geq 0$ we define $U_\eta(s)$ to be the transpose of $L_\eta(s)$.

Theorem 5.1.8 (Kronecker form). *For a pencil $A(s)$ over a field \mathbf{F} , there exist nonsingular matrices P and Q over \mathbf{F} such that*

$$PA(s)Q = \text{block-diag}(sI_{m_0} + B; N_{m_1}(s), \dots, N_{m_b}(s); L_{\varepsilon_1}(s), \dots, L_{\varepsilon_c}(s); U_{\eta_1}(s), \dots, U_{\eta_d}(s)), \quad (5.1)$$

where

$$m_1 \geq \dots \geq m_b \geq 1, \quad \varepsilon_1 \geq \dots \geq \varepsilon_c \geq 0, \quad \eta_1 \geq \dots \geq \eta_d \geq 0,$$

and B is an $m_0 \times m_0$ matrix over \mathbf{F} . The indices, $m_0; b, m_1, \dots, m_b; c, \varepsilon_1, \dots, \varepsilon_c; d, \eta_1, \dots, \eta_d$, are uniquely determined. Denoting $r = \text{rank } A$ and using $\delta_k(A)$ ($k = 0, 1, 2, \dots$) in (5.3), we have

$$b = r - \max_{k \geq 0} \delta_k(A), \quad c = |\text{Col}(A)| - r, \quad d = |\text{Row}(A)| - r, \quad (5.2)$$

$$m_0 = \delta_r(A) - \sum_{i=1}^c \varepsilon_i - \sum_{j=1}^d \eta_j, \quad (5.3)$$

$$m_k = \delta_{r-k}(A) - \delta_{r-k+1}(A) + 1 \quad (k = 1, \dots, b). \quad (5.4)$$

For a regular pencil, in particular, the indices, $m_0; b, m_1, \dots, m_b$, are determined by $\delta_k(A)$ ($k = 0, 1, 2, \dots$).

*Proof.*¹ The formulas (5.2)–(5.4) can be derived from the block diagonal structure (5.1). The uniqueness of the indices $\varepsilon_1, \dots, \varepsilon_c$ and η_1, \dots, η_d is not difficult to establish.

The existence of block diagonal form (5.1) is proven here. Put $A(s) = sX + Y$, where X and Y are matrices over \mathbf{F} .

Claim: There exist nonsingular matrices \bar{P} and \bar{Q} over \mathbf{F} and partitions $(R_1, \dots, R_\mu; R_\infty)$ and $(C_1, \dots, C_\mu; C_\infty)$ of the row set R and the column set C of $\bar{A}(s) = s\bar{X} + \bar{Y} = \bar{P}(sX + Y)\bar{Q}$ such that

$$\begin{aligned} \text{rank } \bar{X}[R_i, C_j] &= 0 & (1 \leq j \leq \mu, j \leq i \leq \infty), \\ \text{rank } \bar{Y}[R_i, C_j] &= 0 & (1 \leq j \leq \mu, j + 1 \leq i \leq \infty), \\ \text{rank } \bar{X}[R_{j-1}, C_j] &= |C_j| & (2 \leq j \leq \mu), \\ \text{rank } \bar{X}[R_\infty, C_\infty] &= |C_\infty|, \\ \text{rank } \bar{Y}[R_i, C_i] &= |R_i| & (1 \leq i \leq \mu). \end{aligned}$$

Here R_μ , R_∞ , and C_∞ can be empty, whereas other blocks are nonempty. Note that $\bar{A}(s)$ is an upper block-triangular matrix and that the rank conditions imply

$$|C_1| \geq |R_1| \geq |C_2| \geq |R_2| \geq \dots \geq |C_\mu| \geq |R_\mu|, \quad |R_\infty| \geq |C_\infty|.$$

The upper block-triangular form in the claim can be constructed as follows. The column set C_1 is determined by a column-transformation for X , since $\bar{X}[R, C_1] = O$ and $\bar{X}[R, C \setminus C_1]$ is of full-column rank by the rank conditions. Then the row set R_1 is determined by a row-transformation for the submatrix $Y[R, C_1]$, since $\bar{Y}[R \setminus R_1, C_1] = O$ and $\bar{Y}[R_1, C_1]$ is of full-row rank. Next, C_2 is determined by a column-transformation for the submatrix $X[R \setminus R_1, C \setminus C_1]$ (with X denoting the modified X), since $\bar{X}[R \setminus R_1, C_2] = O$ and $\bar{X}[R \setminus R_1, (C \setminus C_1) \setminus C_2]$ is of full-column rank. Then the row set R_2 is determined from the submatrix $Y[R \setminus R_1, C_2]$. Continuing this way, we eventually arrive at $C_{\mu+1} = \emptyset$ for some $\mu \geq 0$;

「 $C_{\mu+1} = \emptyset$ 」は「 $\bar{X}[R \setminus \bigcup_{i=1}^\mu R_i, C \setminus \bigcup_{j=1}^\mu C_j]$ は列フルランク」と言い換えた方が分かりやすい。

then we terminate by defining R_∞ and C_∞ to be the complements of $\bigcup_{i=1}^\mu R_i$ and $\bigcup_{j=1}^\mu C_j$, respectively.

In the above claim we may further assume (see **Section 5.1.4**) that

$$\begin{aligned} \bar{X}[R_i, C_j] &= O & \text{unless } 2 \leq j = i + 1 \leq \mu \text{ or } i = j = \infty, \\ \bar{Y}[R_i, C_j] &= O & \text{unless } 1 \leq i = j \leq \infty, \\ \bar{X}[R_{j-1}, C_j] &= \begin{bmatrix} I_{|C_j|} \\ O \end{bmatrix} & (2 \leq j \leq \mu), \end{aligned}$$

¹ The present proof, valid for an arbitrary \mathbf{F} , is communicated by S. Iwata. See Gantmacher [87, Chap. XII, §2] for an alternative proof in the case of $\mathbf{F} = \mathbf{C}$.

$$\begin{aligned}\bar{X}[R_\infty, C_\infty] &= \begin{bmatrix} I_{|C_\infty|} \\ O \end{bmatrix}, \\ \bar{Y}[R_i, C_i] &= \begin{bmatrix} I_{|R_i|} & O \end{bmatrix} \quad (1 \leq i \leq \mu),\end{aligned}$$

where I_N denotes the identity matrix of order N . Then the matrix $\bar{A}(s)$ takes the form depicted in Fig. 5.1 for $\mu = 4$.

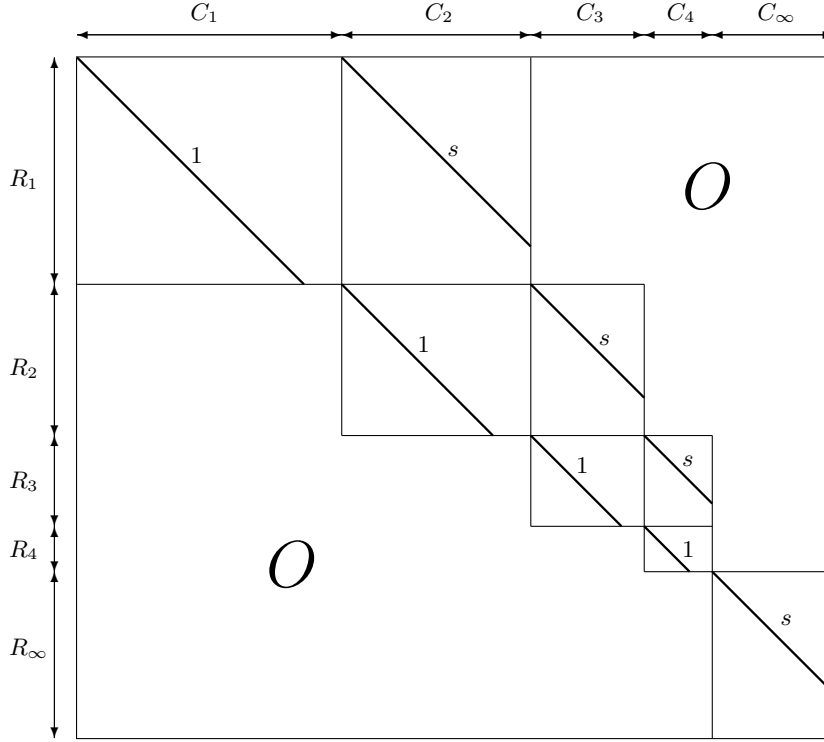


Fig. 5.1. Matrix $\bar{A}(s)$ in the proof for the Kronecker form ($\mu = 4$)

Consider the submatrix $\bar{A}[\bigcup_{i=1}^{\mu} R_i, \bigcup_{j=1}^{\mu} C_j]$. With suitable permutations of rows and columns, it can be put into a block-diagonal form with each diagonal block being equal to $N_m(s)$ with $1 \leq m \leq \mu$ or $L_\varepsilon(s)$ with $0 \leq \varepsilon \leq \mu - 1$, where $N_m(s)$ appears with multiplicity $|R_m| - |C_{m+1}|$ (note: $|C_{\mu+1}| = 0$) and $L_\varepsilon(s)$ with multiplicity $|C_{\varepsilon+1}| - |R_{\varepsilon+1}|$. Namely,

$$\bar{A}\left[\bigcup_{i=1}^{\mu} R_i, \bigcup_{j=1}^{\mu} C_j\right] = \text{block-diag}(N_{m_1}(s), \dots, N_{m_b}(s); L_{\varepsilon_1}(s), \dots, L_{\varepsilon_c}(s))$$

with

$$\begin{aligned}
b &= \sum_{i=1}^{\mu} |R_i| - \sum_{j=2}^{\mu} |C_j|, & c &= \sum_{j=1}^{\mu} |C_j| - \sum_{i=1}^{\mu} |R_i|, \\
|\{k \mid m_k = m\}| &= |R_m| - |C_{m+1}| & (1 \leq m \leq \mu), \\
|\{k \mid \varepsilon_k = \varepsilon\}| &= |C_{\varepsilon+1}| - |R_{\varepsilon+1}| & (0 \leq \varepsilon \leq \mu - 1).
\end{aligned}$$

For the submatrix $\bar{A}[R_{\infty}, C_{\infty}]$, we apply the above argument to its transpose. Let $(\hat{R}_1, \dots, \hat{R}_{\hat{\mu}}; \hat{R}_{\infty})$ and $(\hat{C}_1, \dots, \hat{C}_{\hat{\mu}}; \hat{C}_{\infty})$ be the resulting partitions of R_{∞} and C_{∞} , respectively. Since $\text{rank } \bar{X}[R_{\infty}, C_{\infty}] = |C_{\infty}|$, we have $|\hat{C}_i| = |\hat{R}_{i+1}|$ for $i = 1, \dots, \hat{\mu} - 1$ and $|\hat{C}_{\hat{\mu}}| = 0$. This means that there exist nonsingular matrices \hat{P} and \hat{Q} such that

$$\hat{P}\bar{A}[R_{\infty}, C_{\infty}]\hat{Q} = \text{block-diag}(U_{\eta_1}(s), \dots, U_{\eta_d}(s); s\hat{X}_{\infty} + \hat{Y}_{\infty}),$$

where \hat{X}_{∞} is nonsingular. Finally, we can transform \hat{X}_{∞} to the identity matrix to obtain the desired block-diagonal form (5.1). \blacksquare

5.1.4 Detail of the Second Half of the Proof

276 頁の上の方の Claim 中の条件

$$\begin{aligned} \text{rank } \bar{X}[R_i, C_j] &= 0 & (1 \leq j \leq \mu, j \leq i \leq \infty), \\ \text{rank } \bar{Y}[R_i, C_j] &= 0 & (1 \leq j \leq \mu, j+1 \leq i \leq \infty), \\ \text{rank } \bar{X}[R_{j-1}, C_j] &= |C_j| & (2 \leq j \leq \mu), \\ \text{rank } \bar{X}[R_\infty, C_\infty] &= |C_\infty|, \\ \text{rank } \bar{Y}[R_i, C_i] &= |R_i| & (1 \leq i \leq \mu) \end{aligned}$$

を満たす \bar{X}, \bar{Y} から始めて, 276 頁の下の方の条件:

$$\begin{aligned} \bar{X}[R_i, C_j] &= O & \text{unless } 2 \leq j = i+1 \leq \mu \text{ or } i = j = \infty, \\ \bar{Y}[R_i, C_j] &= O & \text{unless } 1 \leq i = j \leq \infty, \\ \bar{X}[R_{j-1}, C_j] &= \begin{bmatrix} I_{|C_j|} \\ O \end{bmatrix} & (2 \leq j \leq \mu), \\ \bar{X}[R_\infty, C_\infty] &= \begin{bmatrix} I_{|C_\infty|} \\ O \end{bmatrix}, \\ \bar{Y}[R_i, C_i] &= [I_{|R_i|} \ O] & (1 \leq i \leq \mu), \end{aligned}$$

を満たす \bar{X}, \bar{Y} を構成する. 以下, $\mu = 4$ とする.

1. \bar{X}, \bar{Y} は Claim の条件を満たすとす. これを図 5.2 に示す (\bar{Y} を左側に置く. \equiv は行フルランクを, $\equiv\equiv\equiv$ は列フルランクを表わす).

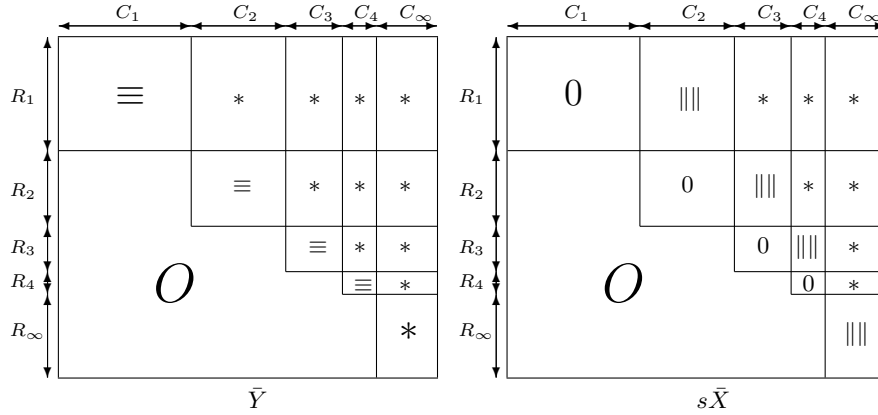


Fig. 5.2. Proof for the Kronecker form (1); initial form

2. $\bar{Y}[R_1, C_1]$ に着目する. 行フルランクなので, $\bar{Y}[R_1, C_1] = [I_{|R_1|} \ O]$ (階数標準形) に変形できる. 全体は図 5.3 のようになる. \bar{X} の形は保存されている.

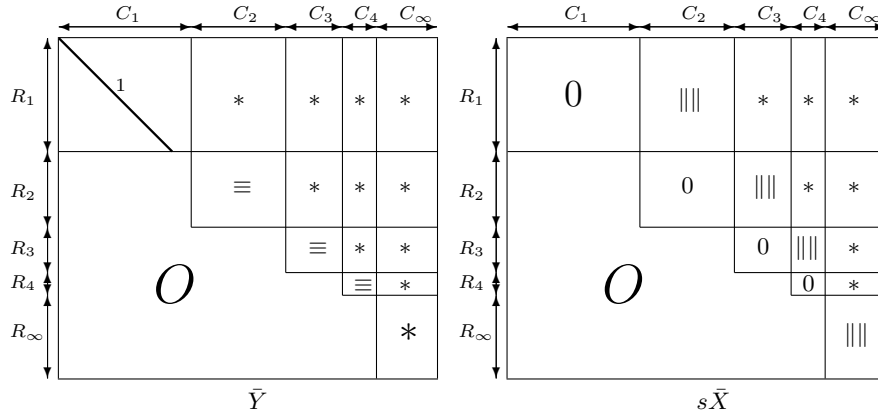


Fig. 5.3. Proof for the Kronecker form (2)

3. $\bar{X}[R_1, C_2]$ に着目する. 列フルランクなので, $\bar{X}[R_1, C_2] = \begin{bmatrix} I_{|C_2|} \\ O \end{bmatrix}$ (階数標準形) に変形できる. このときに R_1 の部分に行変形をするので $\bar{Y}[R_1, C_1]$ の形が壊れるが, C_1 の部分に逆変換の列変形を施す (R_1 の部分の行変形を表わす行列を P として $\begin{bmatrix} P^{-1} & O \\ O & I \end{bmatrix}$ を右から掛ける) と, $\bar{Y}[R_1, C_1]$ の形が元に戻る. その結果, 全体は図 5.4 のようになる.

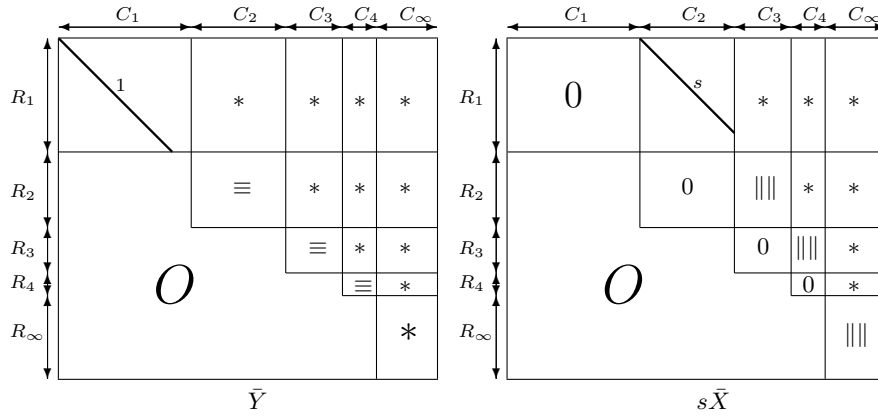


Fig. 5.4. Proof for the Kronecker form (3)

4. $\bar{Y}[R_2, C_2]$ に着目する. 行フルランクなので, $\bar{Y}[R_2, C_2] = [I_{|R_2|} \ O]$ (階数標準形)に変形できる. このときに C_2 の部分に列変形をするので $\bar{X}[R_1, C_2]$ の形が壊れてしまうが, R_1 の部分に逆変換の行変形を施す (C_2 の部分の列変形を表わす行列を Q として $\begin{bmatrix} Q^{-1} & O \\ O & I \end{bmatrix}$ を左から掛ける)と, $\bar{X}[R_1, C_2]$ の形が元に戻る. このときに R_1 の部分に行変形をするので $\bar{Y}[R_1, C_1]$ の形が壊れてしまうが, C_1 の部分に逆変換の列変形を施す ($\begin{bmatrix} Q & O \\ O & I \end{bmatrix}$ を右から掛ける)と, $\bar{Y}[R_1, C_1]$ の形が元に戻る. 全体は図 5.5 のようになる.

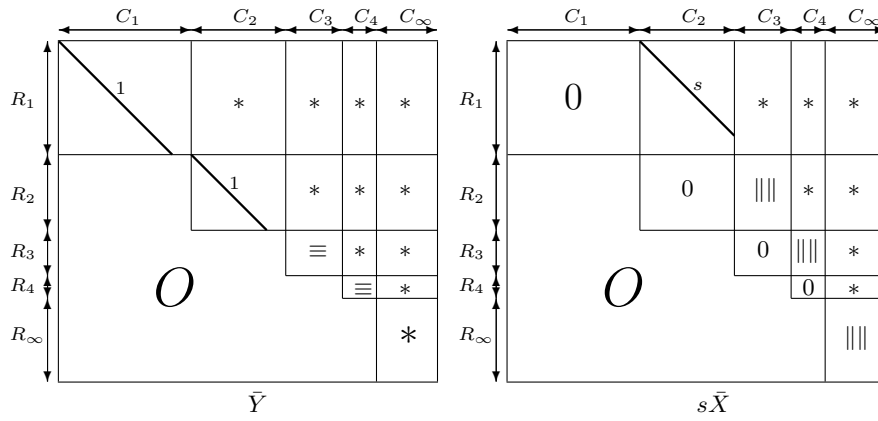


Fig. 5.5. Proof for the Kronecker form (4)

5. このような変換を続けると、図 5.6 のようになる。 \bar{Y} に含まれる I の部分と $s\bar{X}$ に含まれる sI の部分が実現できている。残るは、右上の非零成分を消すことである。

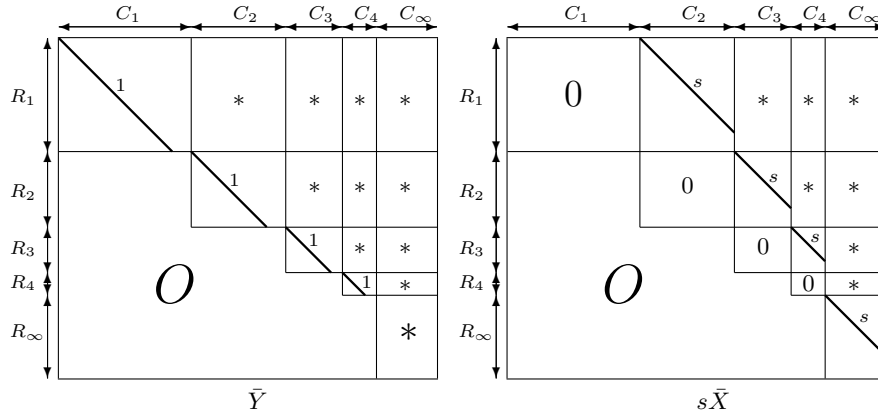


Fig. 5.6. Proof for the Kronecker form (5)

6. (R_4 の行) $\bar{X}[R_\infty, C_\infty]$ が単位行列を含むことに着目して、行変形で $\bar{X}[R_4, C_\infty]$ を O にする。次に、 $\bar{Y}[R_4, C_4]$ が単位行列を含むことに着目して、列変形で $\bar{Y}[R_4, C_\infty]$ を O にする。このとき、 $\bar{X}[R_4, C_4] = O$ なので $\bar{X}[R_4, C_\infty]$ は O のままであり、図 5.7 のようになる。

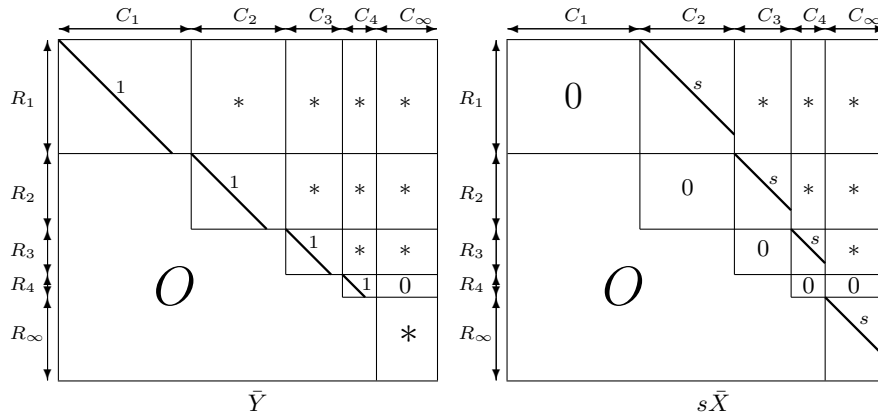


Fig. 5.7. Proof for the Kronecker form (6)

7. (R_3 の行) $\bar{X}[R_\infty, C_\infty]$ が単位行列を含むことに着目して、行変形で $\bar{X}[R_3, C_\infty]$ を O にする。次に、 $\bar{Y}[R_3, C_3]$ が単位行列を含むことに着目し

て、列変形で $\bar{Y}[R_3, C_4 \cup C_\infty]$ を O にする。このとき、 $\bar{X}[R_3, C_3] = O$ なので $\bar{X}[R_3, C_4 \cup C_\infty]$ は元のままであり、図 5.8 のようになる。

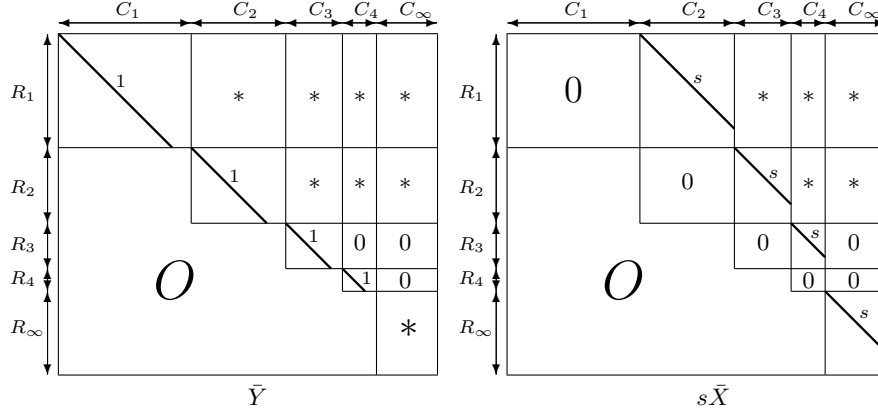


Fig. 5.8. Proof for the Kronecker form (7)

8. (R_2 の行) $\bar{X}[R_3 \cup R_\infty, C_4 \cup C_\infty]$ が単位行列を含むことに着目して、行変形で $\bar{X}[R_2, C_4 \cup C_\infty]$ を O にする。次に、 $\bar{Y}[R_2, C_2]$ が単位行列を含むことに着目して、列変形で $\bar{Y}[R_2, C_3 \cup C_4 \cup C_\infty]$ を O にする。このとき、 $\bar{X}[R_2, C_2] = O$ なので $\bar{X}[R_2, C_3 \cup C_4 \cup C_\infty]$ は元のままであり、図 5.9 のようになる。

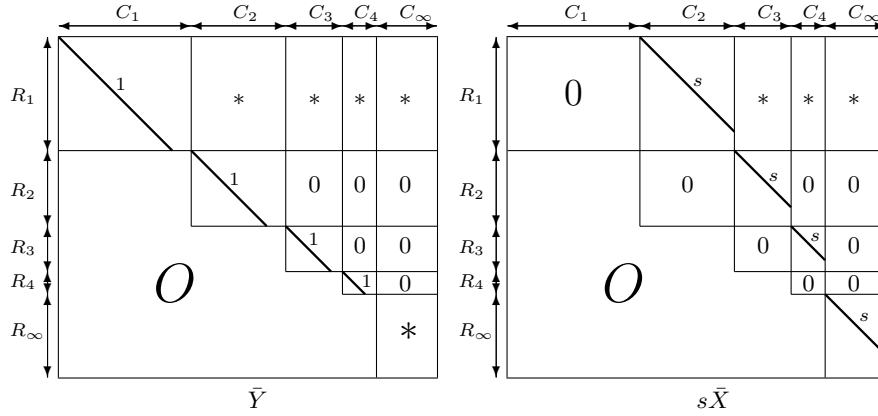


Fig. 5.9. Proof for the Kronecker form (8)

9. (R_1 の行) $\bar{X}[R_2 \cup R_3 \cup R_\infty, C_3 \cup C_4 \cup C_\infty]$ が単位行列を含むことに着目して、行変形で $\bar{X}[R_1, C_3 \cup C_4 \cup C_\infty]$ を O にする。次に、 $\bar{Y}[R_1, C_1]$

が単位行列を含むことに着目して、列変形で $\bar{Y}[R_1, C_2 \cup C_3 \cup C_4 \cup C_\infty]$ を O にする。このとき、 $\bar{X}[R_1, C_1] = O$ なので $\bar{X}[R_1, C_2 \cup C_3 \cup C_4 \cup C_\infty]$ は元のままであり、図 5.10 のようになる。これは求める形になっている。(END)

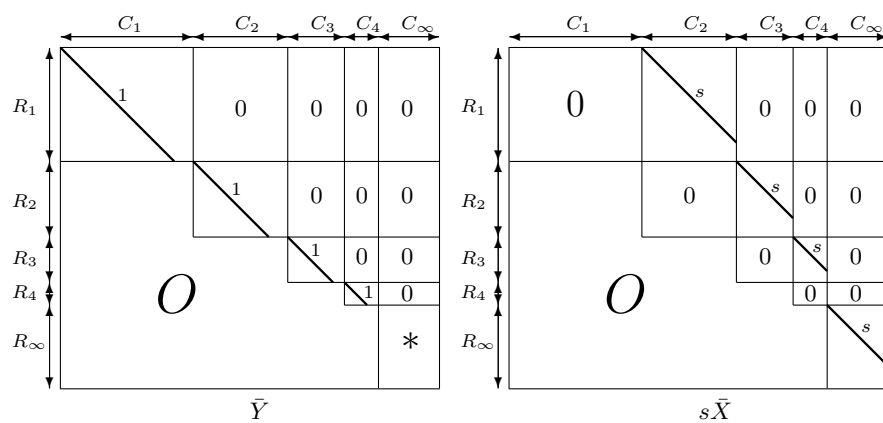


Fig. 5.10. Proof for the Kronecker form (9); the desired form