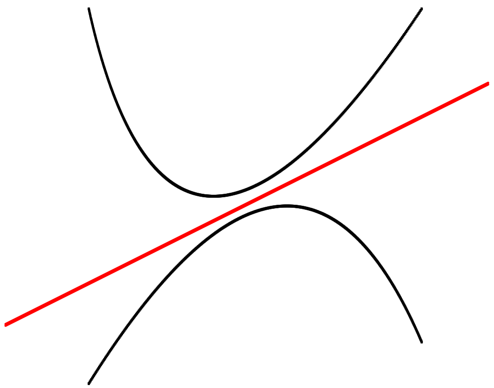
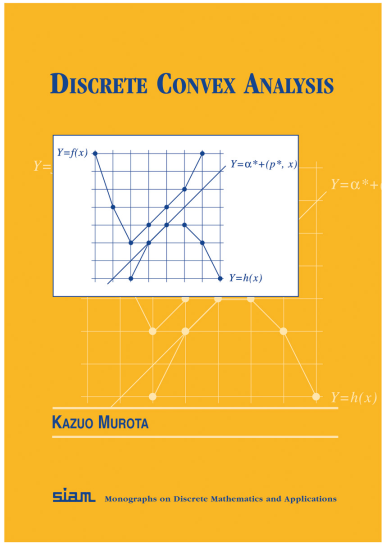
Modern Aspects of Submodularity, Georgia Tech, March 19, 2012

Minimization and Maximization Algorithms in Discrete Convex Analysis

Kazuo Murota (U. Tokyo)





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Contents

B1.

Submodularity and Convexity (2000's)

Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms (Grötschel-Lovász-Schrijver/ Jensen-Korte, Lovász)

  min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

  set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

  discrete separation, Fenchel min-max

Duality for submodular set functions = Convexity + Discreteness

On the other hand ...

decreasing
marginal return $\longleftrightarrow$ concave/submodular

This means ...

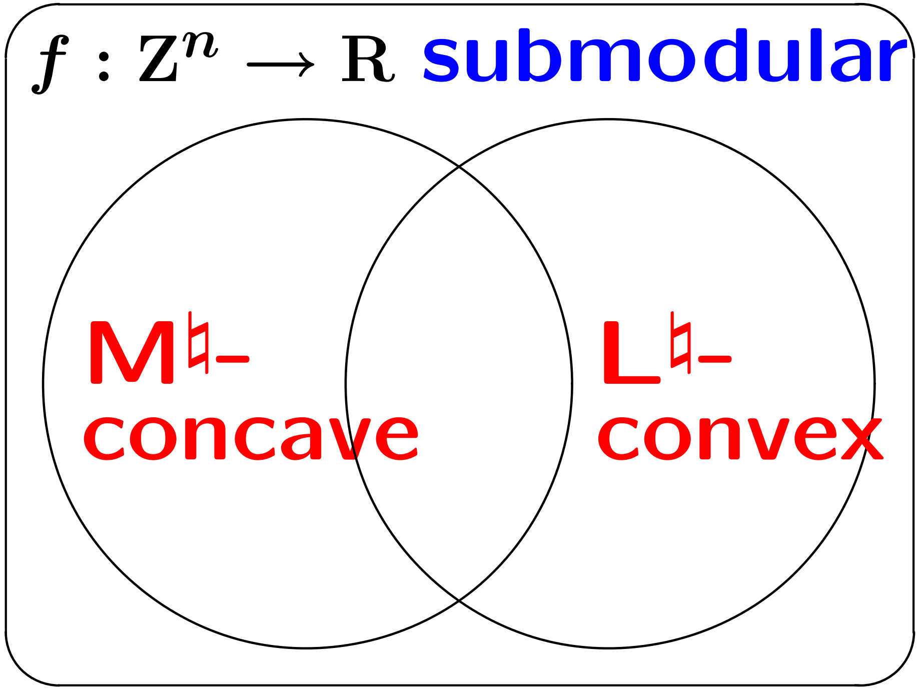
Submodular $\approx$ Concave

Moreover ...

$\rho(X) = \varphi(|X|)$ (φ: concave) is submodular

Submodularity & Convexity in DCA

- $\mathrm{M}^\natural$-concave function is submodular
- $\mathrm{L}^\natural$-convex function is submodular



sbm + sbm ⇒ sbm

$\mathrm{L}^\natural + \mathrm{L}^\natural \Rightarrow \mathrm{L}^\natural$

$\mathrm{M}^\natural + \mathrm{M}^\natural \not\Rightarrow \mathrm{M}^\natural$

- Sum of $\mathrm{M}^\natural$-concave fns is submodular

(i) $\mathrm{M}^\natural$ (ii) $\mathrm{M}^\natural + \mathrm{M}^\natural$ (iii) $\mathrm{M}^\natural + \mathrm{M}^\natural + \mathrm{M}^\natural$

B2.

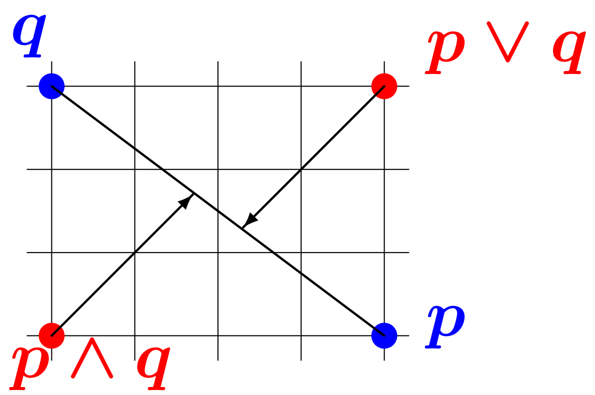
L-convex and M-convex Functions

L-convex Function (L = Lattice)

$g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

$p \vee q$ compnt-max

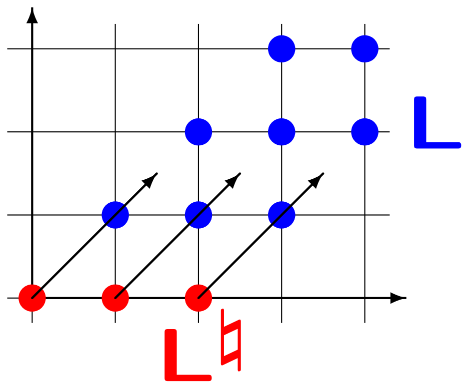
$p \wedge q$ compnt-min



Def: g is L-convex $\iff$

- Submodular: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$
- Translation: $\exists r,\ \forall p$: $g(p + 1) = g(p) + r$

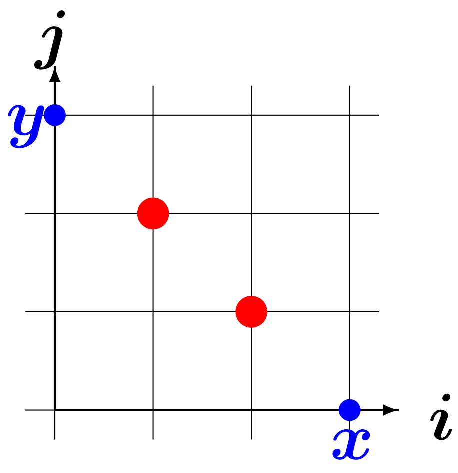
$$\mathrm{L}_{n+1} \simeq \mathrm{L}^{\natural}_{n} \supsetneqq \mathrm{L}_{n}$$



M-convex Function (M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector

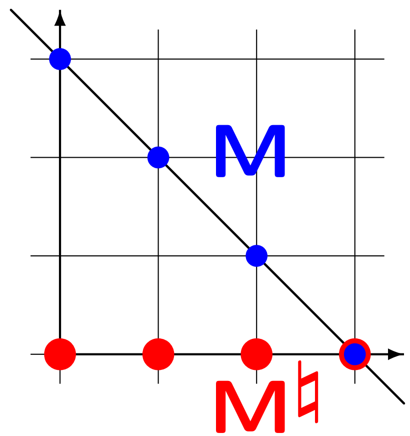


Def: f is M-convex

$\iff \quad \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom} f \subseteq$ const-sum hyperplane



$$\mathrm{M}_{n+1} \simeq \mathrm{M}_n^{\natural} \supsetneqq \mathrm{M}_n$$

A.
Minimization (General)

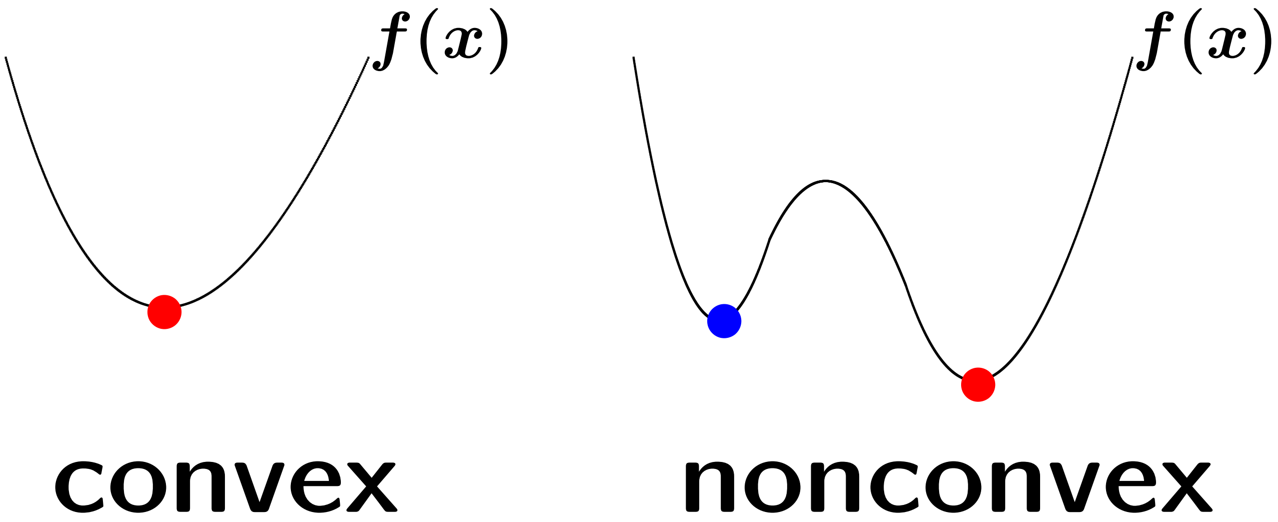
Optimality Criterion

Descent Method

Scaling and Proximity

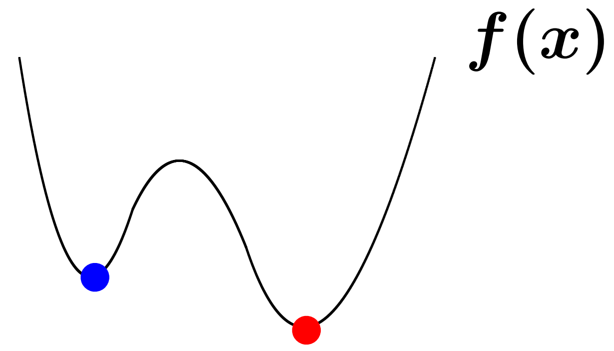
Optimality Criterion

Global opt vs Local opt



Local opt wrt neighborhood

Descent Method



S0: Initial sol x^*

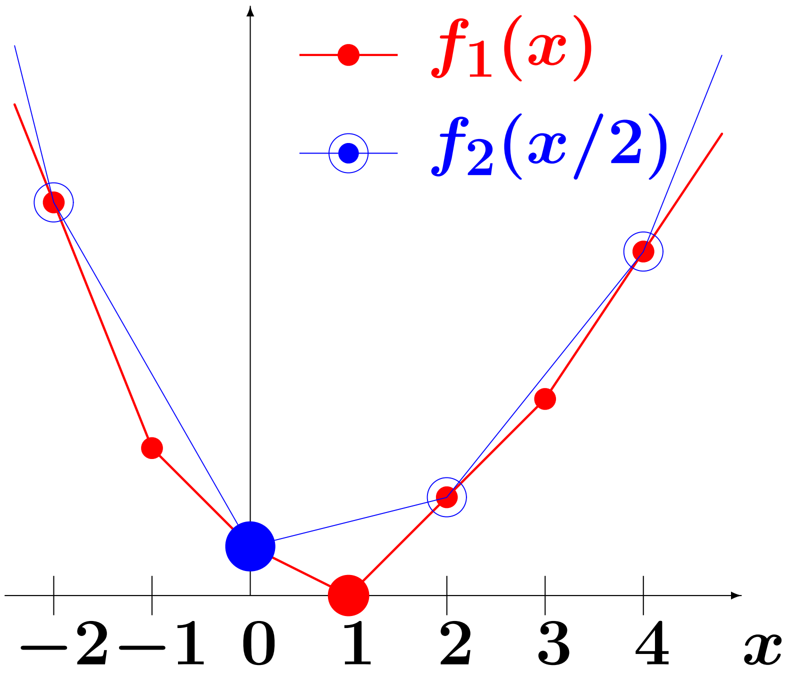
S1: Minimize $f(x)$ in nbhd of x^* to obtain $x^\bullet$

S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (local opt)

S3: Update $x^* = x^\bullet$; go to S1

$\cdots\cdots$What is nbhd ?

Scaling and Proximity



Proximity theorem:

True minimum ● exists

in a neighborhood of

a scaled local minimum ●

$\Rightarrow$ efficient algorithm

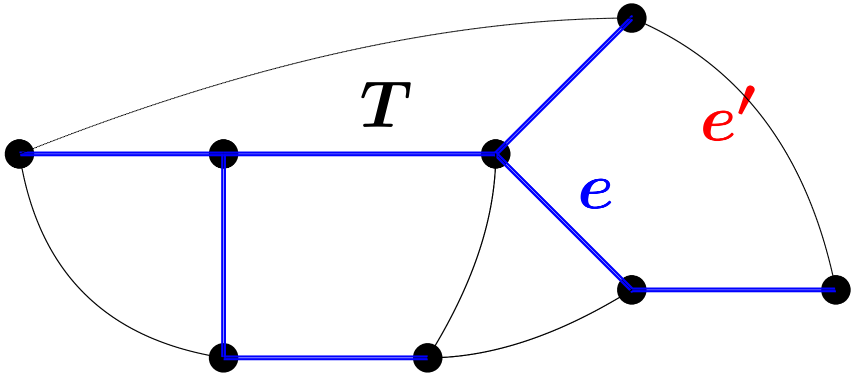
Facts in DCA:

- Scaling preserves L-convexity
- Scaling does NOT preserve M-convexity
- Proximity thms known for L-conv and M-conv

A1.

M-convex Minimization

Min Spanning Tree Problem



length $d : E \to \mathrm{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: MST $\iff \tilde{d}(T) \leq \tilde{d}(T - e + e')$

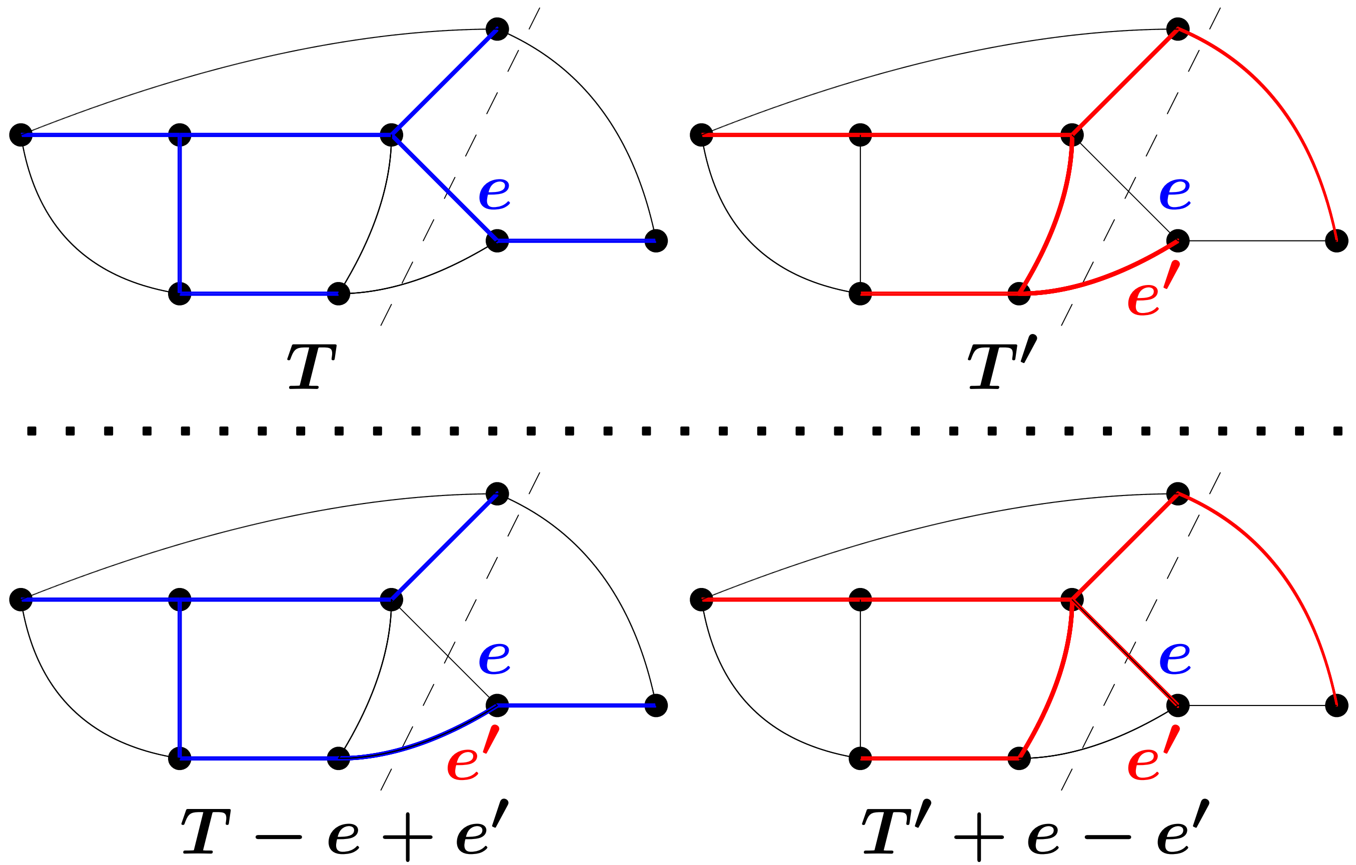
$\iff d(e) \leq d(e')$ if $T - e + e'$ is tree

Algorithm Kruskal's, Kalaba's

DCA view

- linear optimization on an M-convex set
- M-optimality: $f(x^*) \leq f(x^* - e_i + e_j)$

Tree: Exchange Property



Given pair of trees

New pair of trees

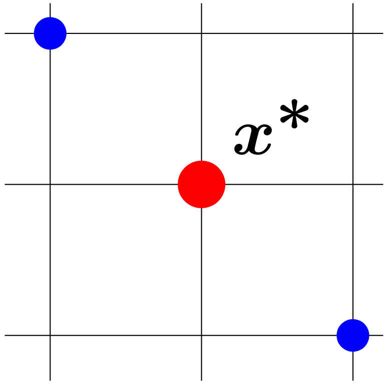
Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Local vs Global Opt (M-conv)

Thm : $f : \mathbb{Z}^n \to \mathbb{R}$ M-convex (Murota 96)

x^*: global opt

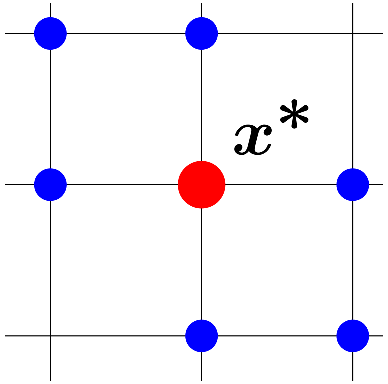
$\iff$ local opt $f(x^*) \le f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M♮-convex fn $\Rightarrow$



Steepest Descent for M-convex Fn

(Murota 03, Shioura 98, 03, Tamura 05)

S0: Find a vector $x \in \mathrm{dom} f$

S1: Find $i \neq j$ that minimize $f(x - e_i + e_j)$

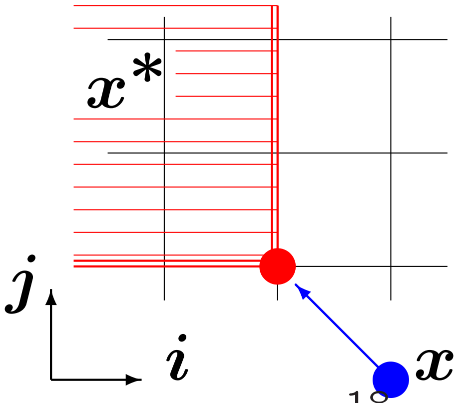
S2: If $f(x) \leq f(x - e_i + e_j)$, stop ($x$: minimizer)

S3: Set $x := x - e_i + e_j$ and go to S1

Minimizer Cut Thm (Shioura 98)

$\exists$ minimizer x^* with $x^*_i \leq x_i - 1$, $x^*_j \geq x_j + 1$

- Kalaba's for min spannning tree
- Dress–Wenzel's for valuated matroid



A2.

L-convex Minimization

Shortest Path Problem (one-to-all)

one vertex (s) to all vertices, length $\ell \geq 0$, integer

Dual LP

Maximize	$\Sigma \, p(v)$
subject to	$p(v) - p(u) \leq \ell(u,v) \ \forall (u,v)$
	$p(s) = 0$

Algorithm Dijkstra's

DCA view

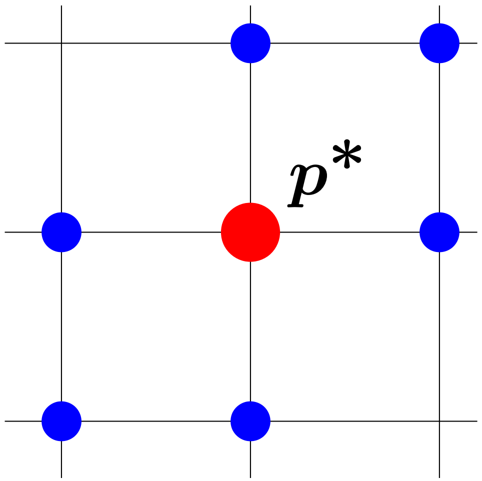
- linear optimization on an L$^\natural$-convex set (in polyhedral description)
- Dijkstra's algorithm (M.-Shioura 12)
  = steepest ascent for L$^\natural$-concave maximization with uniform linear objective $(1, 1, \ldots, 1)$

Local vs Global Opt (L♮-conv)

Thm : $g : \mathbb{Z}^n \to \mathbb{R}$ L♮-convex (Murota 98,03)

p^*: global opt

$\Longleftrightarrow$ local opt $g(p^*) \leq g(p^* \pm q)$ $(\forall q \in \{0,1\}^n)$



Ex: $p^* + (0,1,0,1,1,1,0,0)$

$\Longleftrightarrow$ $\rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$

takes min at $X = \emptyset$

Can check with n^5 (or less) fn evals using submodular fn min algorithm (Iwata-Fleischer-Fujishige, Schrijver, Orlin)

Steepest Descent for L♮-convex Fn

(Iwata 99, Murota 00, 03, Kolmogorov-Shioura 09)

S0: Find a vector $p \in \mathrm{dom}\, g$

S1: Find $\varepsilon = \pm 1$ and X that minimize $g(p + \varepsilon \chi_X)$

S2: If $g(p) \leq g(p + \varepsilon \chi_X)$, stop ($p$: minimizer)

S3: Set $p := p + \varepsilon \chi_X$ and go to S1

- Dijkstra's algorithm for shortest path (M.-Shioura 12)

  π: potential, $V \setminus U$: permanent labeled

  Special case with $g(p) = -1^\top p$:

  $\pi(v) = \min\{p(u) + \ell(u, v) \mid u \notin U\} \quad (v \in U \setminus \{s\})$

Optimality & Proximity Theorems

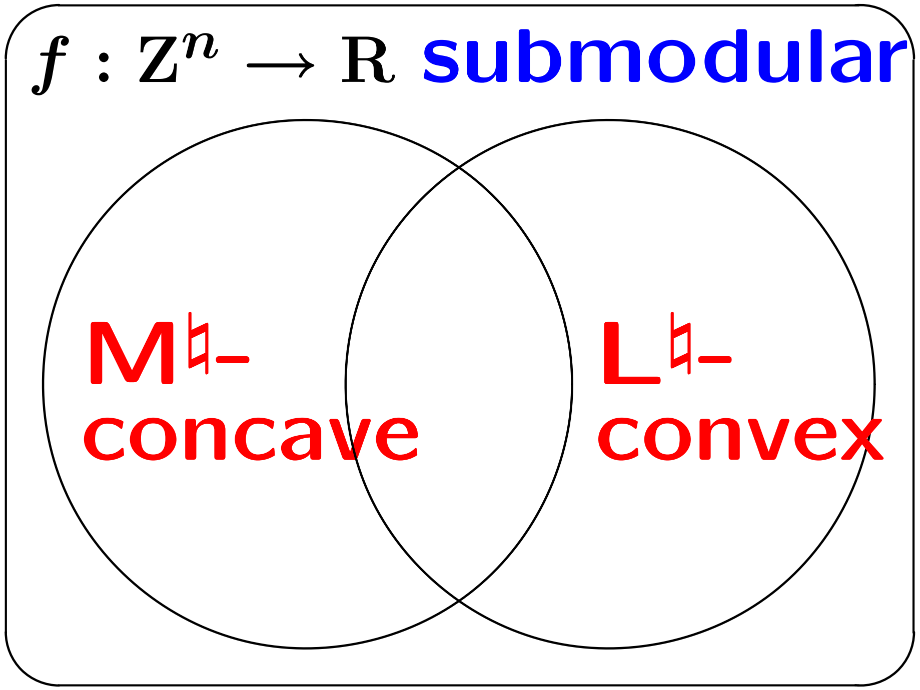
Func Class	Optimality	Proximity
L-convex	$f(x^*) \leq f(x^* + \chi_S) \quad (\forall S)$ $f(x^* + 1) = f(x^*)$ (M. 01)	$\|x^* - x^\alpha\| \leq (n-1)(\alpha-1)$ (Iwata-Shigeno 03)
M-convex	$f(x^*) \leq f(x^* - \chi_u + \chi_v)$ $(\forall u, v \in V)$ (M. 96)	$\|x^* - x^\alpha\| \leq (n-1)(\alpha-1)$ (Moriguchi-M.-Shioura 02)
L2-convex (L∗L convol)	$f(x^*) \leq f(x^* + \chi_S) \ (\forall S)$ $f(x^* + 1) = f(x^*)$	$\|x^* - x^\alpha\| \leq 2(n-1)(\alpha-1)$ (M.-Tamura 04)
M2-convex (M+M)	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W)$ (M. 01)	$\|x^* - x^\alpha\| \leq \frac{n^2}{2}(\alpha-1)$ (M.-Tamura 04)
integrally convex	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W)$ (Favati-Tardella 90)	???

$\| \cdot \| = \| \cdot \|_\infty$

A3.
Submodular Maximization

Submodularity & Convexity in DCA

- $\mathrm{M}^\natural$-concave function is submodular
- $\mathrm{L}^\natural$-convex function is submodular



sbm + sbm $\Rightarrow$ sbm

$\mathrm{L}^\natural + \mathrm{L}^\natural \Rightarrow \mathrm{L}^\natural$

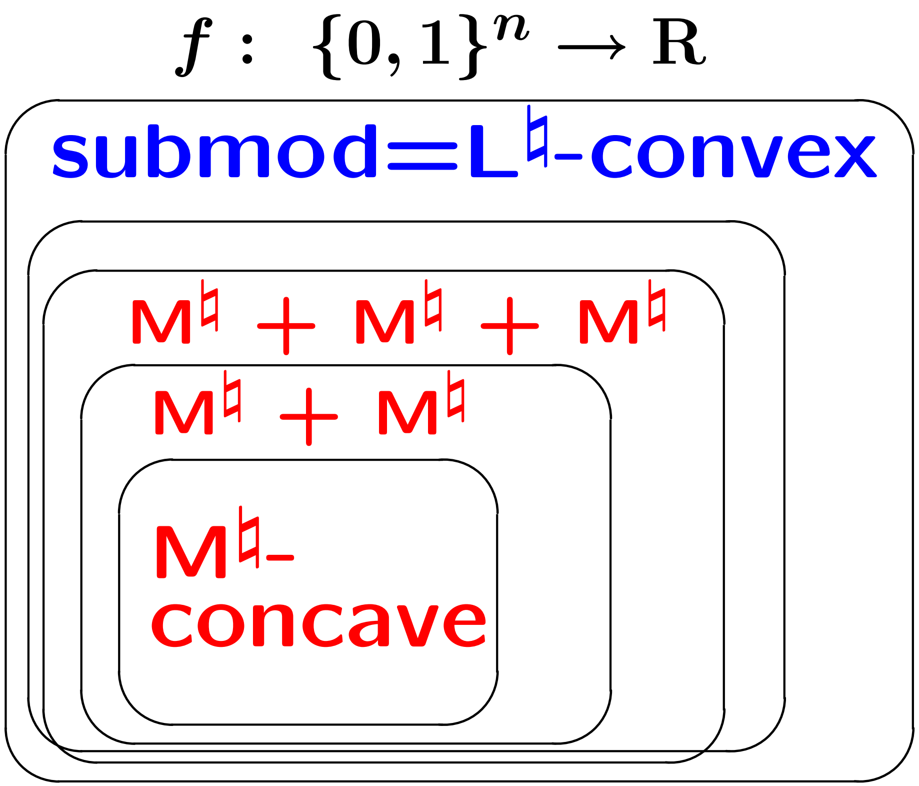
$\mathrm{M}^\natural + \mathrm{M}^\natural \not\Rightarrow \mathrm{M}^\natural$

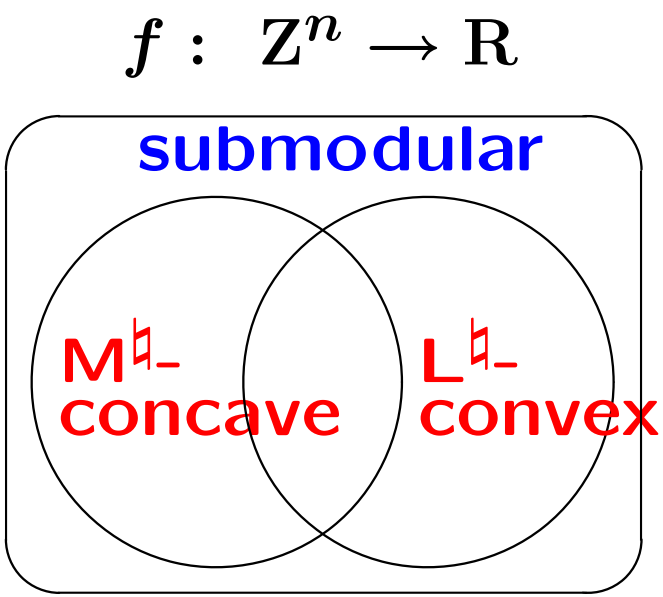
- Sum of $\mathrm{M}^\natural$-concave fns is submodular

(i) $\mathrm{M}^\natural$ (ii) $\mathrm{M}^\natural + \mathrm{M}^\natural$ (iii) $\mathrm{M}^\natural + \mathrm{M}^\natural + \mathrm{M}^\natural$

Submodular Set Function in DCA

- Submodular set func = L$^\natural$-convex on $\{0,1\}^n$
- (Sums of) M$^\natural$-concave form a nice subclass





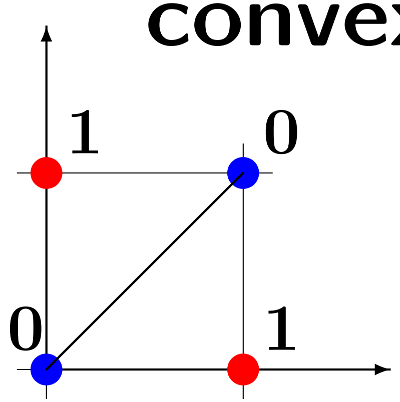
Set Function and Extensions

Set function $\iff$ **Function on** $\{0,1\}^n$

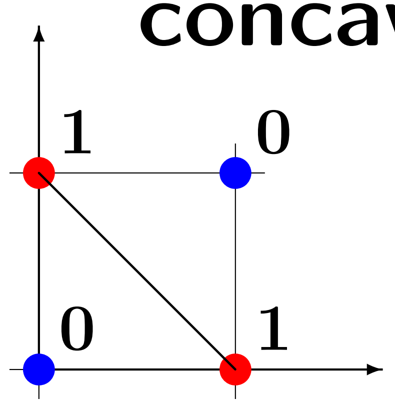
$$\rho(X) \quad = \quad \hat{\rho}(\chi_X)$$

Every set function $\rho : \{0,1\}^n \to \mathbb{R}$ can be extended to convex/concave function

convex ext



concave ext



M♮-concave Set Functions

M$^\natural$-concave is submodular (NOT conversely)

M$^\natural$-concave forms a nice subclass for maximization

- $\mu(X) = \varphi(|X|)$ (φ: concave)
- $\mu(X) = \sum_{A \in \mathcal{T}} \varphi_A(|A \cap X|)$ (φ_A: concave)

  $\mathcal{T}$: laminar ($A, B \in \mathcal{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$)
- max-value $\mu(X) = \max\{a_i \mid i \in X\}$
- matroid rank (Fujishige 05)

  $\mu(X) = \max\{|I| \mid I :$ independent, $I \subseteq X\}$
- weighted matroid rank ($w \geq 0$) (Shioura 09)

  $\mu(X) = \max\{w(I) \mid I :$ independent, $I \subseteq X\}$

Dual Character of Matroid Rank Func

$\rho(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$

is L$^\natural$-convex and M$^\natural$-concave

Self-Conjugacy: $\rho(X) = |X| - \rho^\bullet(\chi_X)$

Prf: $\rho^\bullet(\chi_X) = \max_Y\{|X \cap Y| - \rho(Y)\}$

$= \max_{Y \supseteq X}\{|X \cap Y| - \rho(Y)\} = |X| - \rho(X)$

ρ subm $\Rightarrow$ ρ L$^\natural$-conv $\Rightarrow$ $\rho^\bullet$ M$^\natural$-conv $\Rightarrow$ ρ M$^\natural$-concave

Edmonds's matroid union formula:

$$\max_X\{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y\{\rho_1(Y) + \rho_2(Y) + |V \setminus Y|\}$$

submod maximization (M$^\natural$-concave + M$^\natural$-concave)

submod minimization (L$^\natural$-convex + L$^\natural$-convex)

Polymatroid Rank Function

Polymatroid rank function is NOT M$^\natural$-concave

Example $\rho : 2^V \to \mathbb{Z}$ on $V = \{1, 2, 3, 4\}$ (Shioura)

$\rho(\emptyset) = 0$, $\rho(i) = 2$ $(i \in V)$, $\rho(1, 2) = \rho(3, 4) = 4$,

$\rho(1, 3) = \rho(1, 4) = \rho(2, 3) = \rho(2, 4) = 3$,

$\rho(X) = 4$ if $|X| \geq 3$

Exchange fails for $X = \{1, 2\}$, $Y = \{3, 4\}$

Algorithms for Submodular Set Func

convex extension	concave extension
computable as Lovász extension (Lovász 03)	subclass: $\mathrm{M}^\natural$-concave (valuated matroid) ($\Longrightarrow$ Shioura's talk)

1. Greedy algorithm for max of an $\mathrm{M}^\natural$-concave fn (Dress-Wenzel 90)
2. Matroid intersection-type algorithm for max of a sum of two $\mathrm{M}^\natural$-concave fns (Murota 96)
3. Pipage rounding algorithm for approx max of a sum of several nondecr. $\mathrm{M}^\natural$-concave fns (Shioura 09)

M$^\natural$-concave Maximization Algorithm

$\mu(X)$: M$^\natural$-concave set function ($\mu(\emptyset) > -\infty$)

Greedy algorithm

S0: Put $X := \emptyset$

S1: Find $j \in V \setminus X$ that $\boxed{\text{maximizes } \mu(X \cup \{j\})}$

S2: If $\mu(X) \geq \mu(X \cup \{j\})$, stop ($X$:maximizer of μ)

S3: Set $X := X \cup \{j\}$ and go to S1

- (Variant of) Dress–Wenzel's for valuated matroid
- Kruskal's algorithm for min spannning tree

A4.
M-convex Intersection Algorithm

(Fenchel Duality)

Intersection Problem $(f_1 + f_2)$

Recall: $\mathrm{L}^\natural + \mathrm{L}^\natural \Rightarrow \mathrm{L}^\natural$, $\mathrm{M}^\natural + \mathrm{M}^\natural \not\Rightarrow \mathrm{M}^\natural$

M-convex Intersection Algorithm:

Minimizes $f_1 + f_2$ for M$^\natural$-convex f_1, f_2

$\Leftrightarrow$ Maximizes $f_1 + f_2$ for M$^\natural$-concave f_1, f_2
(submodular function maximization)

$\Leftrightarrow$ Fenchel duality (min $=$ max)

$\Rightarrow$ Valuated matroid intersection (Murota 96)

$\Rightarrow$ Weighted matroid intersection

(Edmonds, Lawler, Iri-Tomizawa 76, Frank 81)

M-concave Intersection: Max [M$^\natural$+M$^\natural$]

[Concave version] M$^\natural$+M$^\natural$ is NOT M$^\natural$

f_1, f_2 : M$^\natural$-concave $(\mathbf{Z}^n \to \mathbf{R})$, $x^* \in \mathrm{dom} f_1 \cap \mathrm{dom} f_2$

$f_1 + f_2$ is submodular, NOT M$^\natural$-concave

(1) x^* maximizes $\boxed{f_1 + f_2}$ (Murota 96)

$\iff$ $\exists\ p$ (certificate of optimality)

- x^* maximizes $\boxed{f_1(x) - \langle p, x \rangle}$ (M-opt thm)
- x^* maximizes $\boxed{f_2(x) + \langle p, x \rangle}$ (M-opt thm)

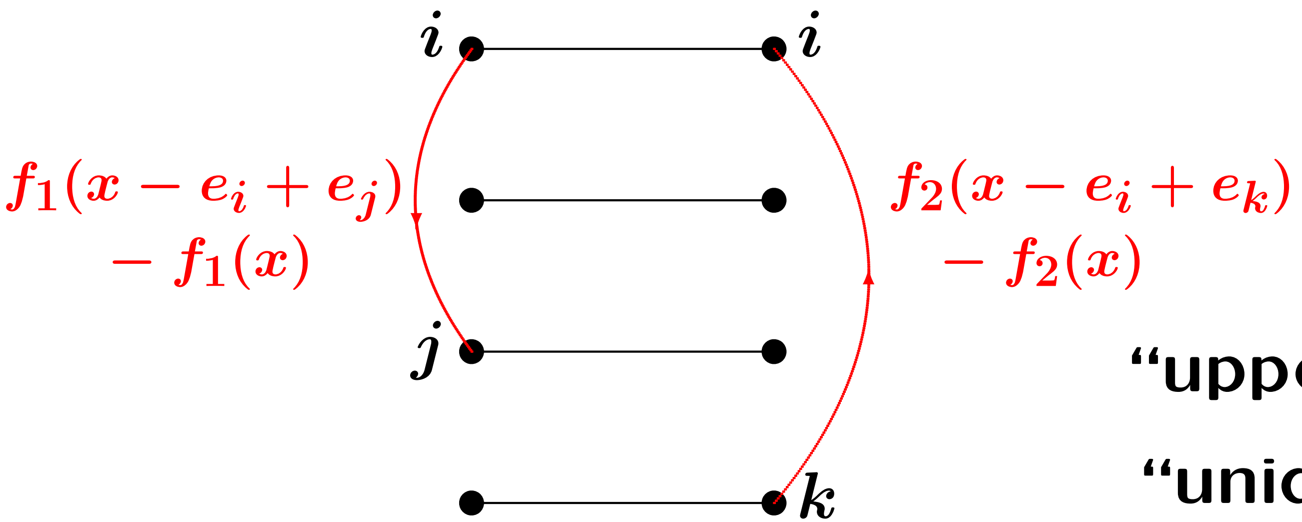
(2) $\mathrm{argmax}\,(f_1 + f_2) = \mathrm{argmax}\,(f_1 - p) \cap \mathrm{argmax}\,(f_2 + p)$

(3) f_1, f_2 are integer-valued $\Rightarrow$ integral p

M-convex Intersection Algorithms

Natural extensions of
weighted (poly)matroid intersection algorithms

Exchange arcs are weighted



"upper-bound lemma"

"unique-max lemma"

- cycle-canceling (Murota 96, 99)
- successive shortest path (Murota-Tamura 03)
- scaling (Iwata-Shigeno 03, Iwata-Moriguchi-M. 05)

Submodular Max. under Matroid Constraint

Maximize $f(S)$ s.t. $|S| = k$

Maximize $f(S)$ s.t. $|S| \leq k$

Maximize $f(S)$ s.t. S: base in a matroid

Maximize $f(S)$ s.t. S: independent in a matroid

Submodular function maximization under a matroid constraint is an NP-hard problem in general

BUT

If f is M$^\natural$-concave, this is an M$^\natural$-concave intersection problem and, hence poly-time solvable

(This slide shows my answer to a question during the talk)

Submodular Welfare Maximization

Maximize $f_1(S_1) + f_2(S_2) + \cdots + f_k(S_k)$ **s.t.**

$(S_1, S_2, \ldots, S_k)$: **partition of** S (f_i: **submodular**)

Submodular welfare maximization is an NP-hard problem in general

BUT

If f_i's are M$^\natural$-concave, this reduces to an M$^\natural$-concave intersection (or, M$^\natural$-concave convolution) and, hence poly-time solvable.

This is the case if $f_i(X) = \varphi_i(|X|)$ with concave φ_i

(This slide shows my answer to a question at the end of the talk)

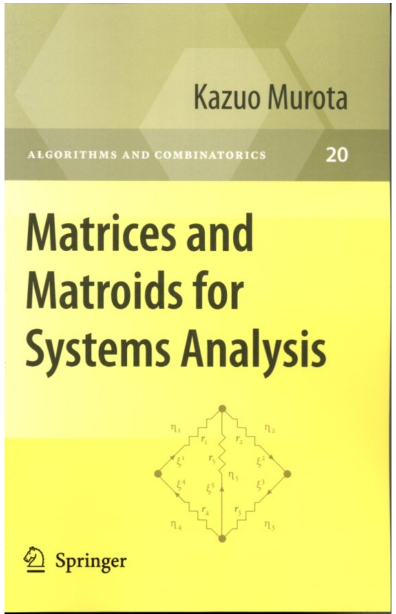
Books

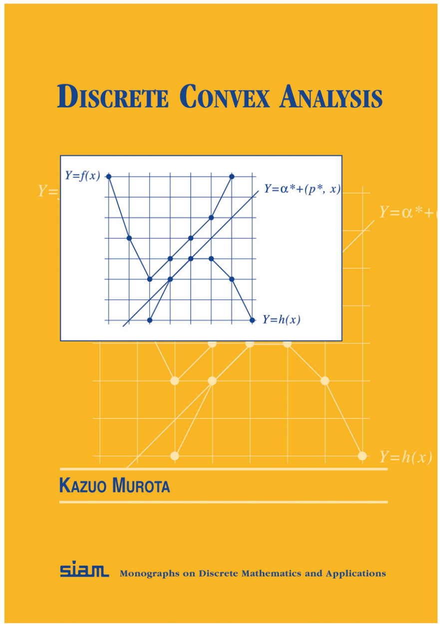
Murota: Matrices and Matroids for Systems Analysis, Springer, 2000/2010 (Chap.5)

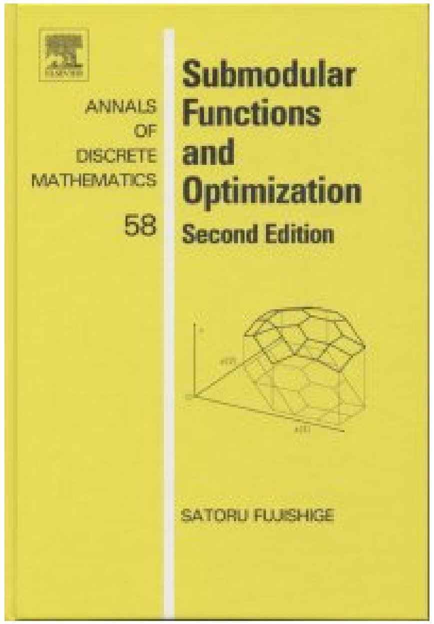
valuated matroid intersection algorithm

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)







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