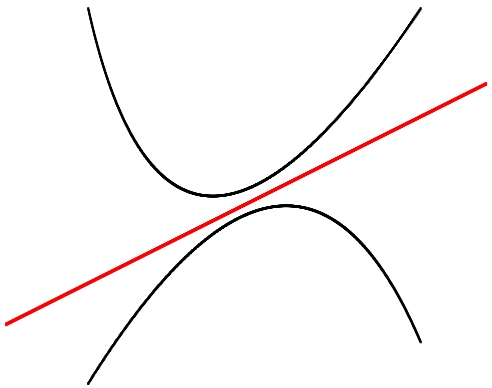
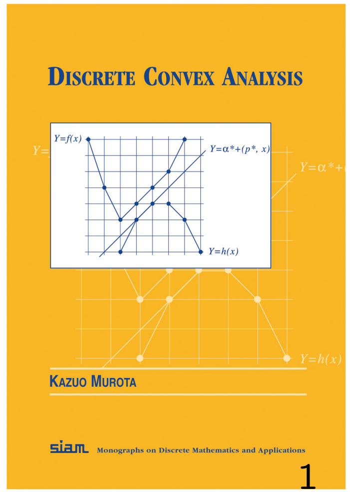
NIPS workshop: Discrete Optimization in Machine Learning:
Connecting Theory and Practice (Lake Tahoe, December 9, 2013)

# Discrete Convex Analysis:

## Basics, DC Programming, and Submodular Welfare Algorithm

**Kazuo Murota (U. Tokyo)**



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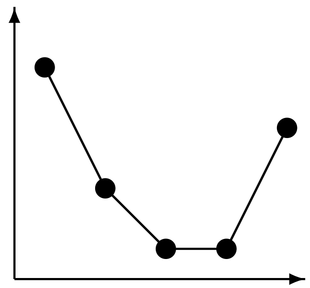
# Discrete Convex Analysis

Convexity Paradigm in Discrete Optimization

## Matroid Theory + Convex Analysis

| | | |
|---|---|---|
| Submodular fn | $\leftrightarrow$ | Matroid base |
| $\Downarrow$ | | $\Downarrow$ |
| L-convex fn | $\leftrightarrow$ | M-convex fn |



- Global optimality $\Longleftrightarrow$ local optimality
- Conjugacy: Legendre–Fenchel transform
- Duality (Fenchel min-max, discrete separation)
- Minimization algorithms
- Applications: OR, game, economics, matrices

# Applications

- Combinatorial optimization
  - matching, even factor, min-cost flow, shortest path, min-cost tension
- Mathematical economics / Game theory
  - Walrasian equilibrium, stable marriage
- Operations research
  - inventory, queueing, resource allocation
- Discrete structures
  - finite metric space
- Algebra
  - polynomial matrix, tropical geometry

# Some History

| | | |
|---|---|---|
| 1935 | Matroid | Whitney, Nakasawa |
| 1965 | Submodular function | Edmonds |
| 1975 | Application of matroid | Iri, Recski |
| 1983 | Submodularity and convexity | Lovász, Frank, Fujishige |
| 1990 | Valuated matroid | Dress–Wenzel |
| | Integrally convex fn | Favati–Tardella |
| 1996 | Discrete convex analysis | Murota |
| 2000 | Submod. fn minimization algorithm | Iwata–Fleischer–Fujishige, Schrijver |

# Contents

B1. Submodularity and Convexity

B2. L-convex and M-convex Functions

B3. Conjugacy — Legendre transform

B4. Duality

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**Recent Topics** (with T. Maehara)

T1. Discrete DC Programming

↖ use of conjugacy

T2. Valuated Matroid-Based Submodular Welfare Algorithm

↖ use of duality

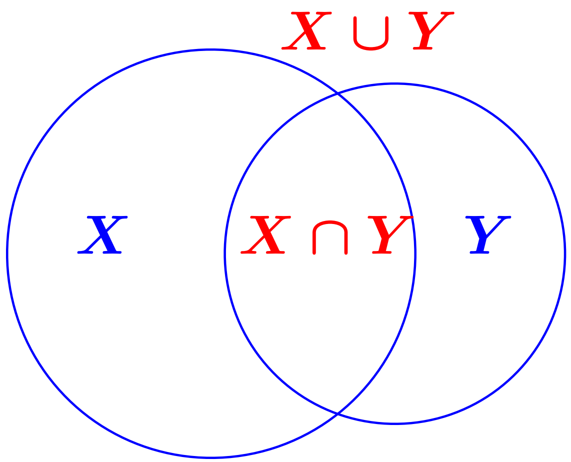
# B1.

## Submodularity and Convexity

# Submodular Function

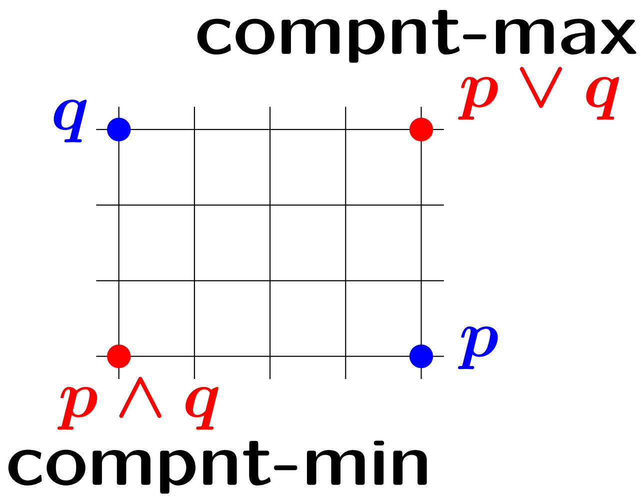
**Set function $\rho$ is submodular:**

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$




$g : \mathrm{Z}^n \to \mathrm{R}$ **is submodular:**

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$



# Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

(Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

discrete separation, Fenchel min-max

**Duality for submodular set functions = Convexity + Discreteness**

# Frank's Discrete Separation

(Frank 82)

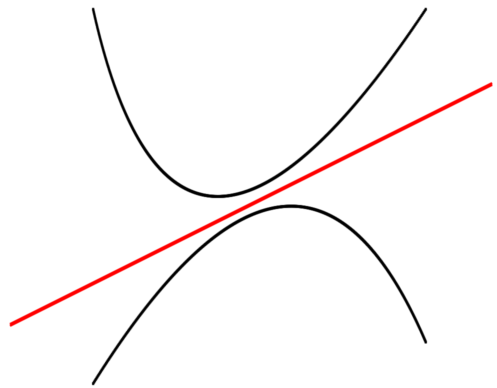
$\rho : 2^V \to \mathrm{R}$: **submodular** $(\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathrm{R}$: **supermodular** $(\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathrm{R}^V$:

  $$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- $\rho$, $\mu$: **integer-valued** $\Rightarrow x^* \in \mathrm{Z}^V$



Equivalent to Edmonds' polymatroid intersection

## On the other hand ...

decreasing
marginal return $\longleftrightarrow$ concave/submodular

## This means ...

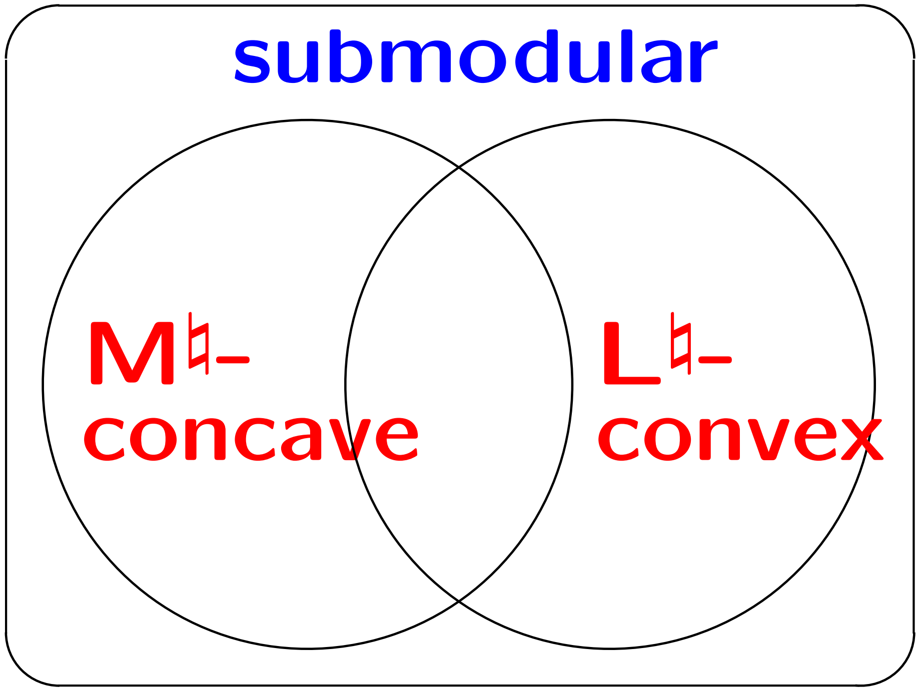
**Submodular $\approx$ Concave**

## Moreover ...

$\rho(X) = \varphi(|X|)$ ($\varphi$: concave) is submodular

# Submodularity & Convexity in DCA

- M$^\natural$-concave function is submodular
- L$^\natural$-convex function is submodular

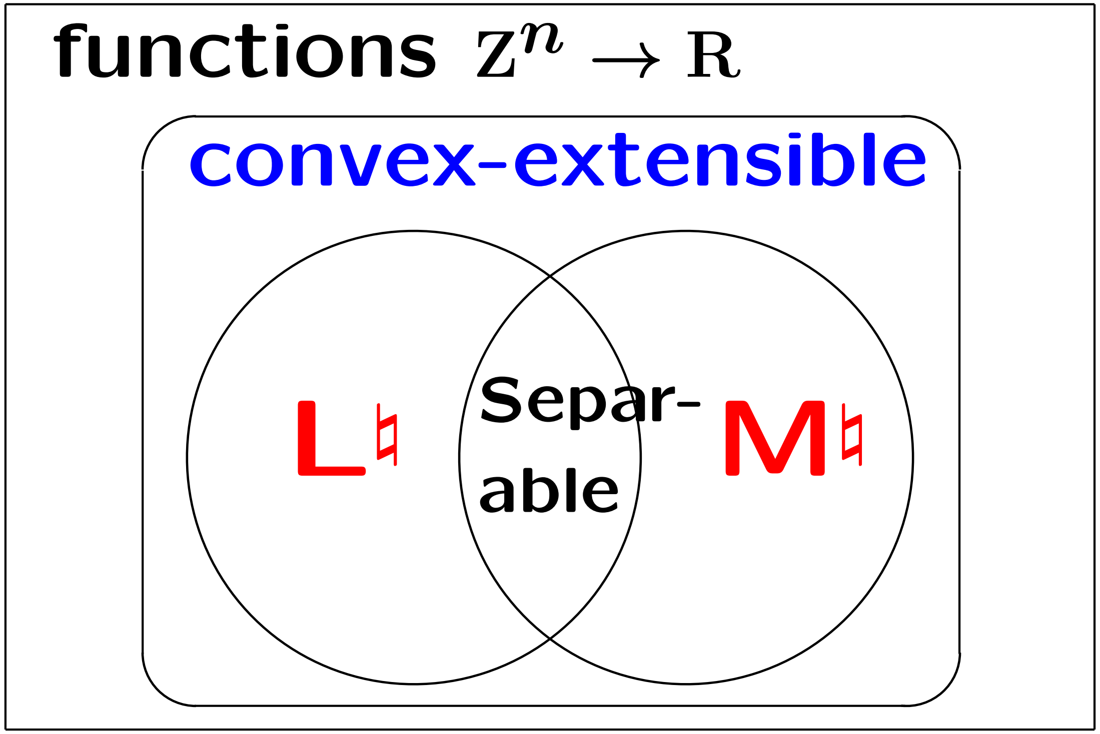



- Sum of M$^\natural$-concave fns is submodular
- Sum of L$^\natural$-convex fns is L$^\natural$-conv (subm)

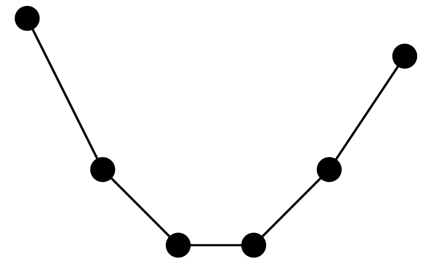
# B2.

## L-convex and M-convex Functions

# Discrete Convex Functions




$$f : Z^n \to R$$

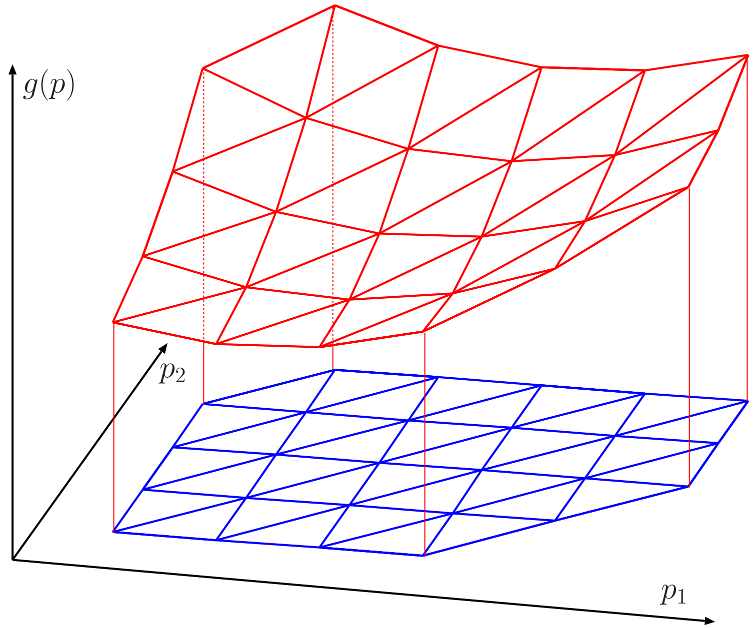


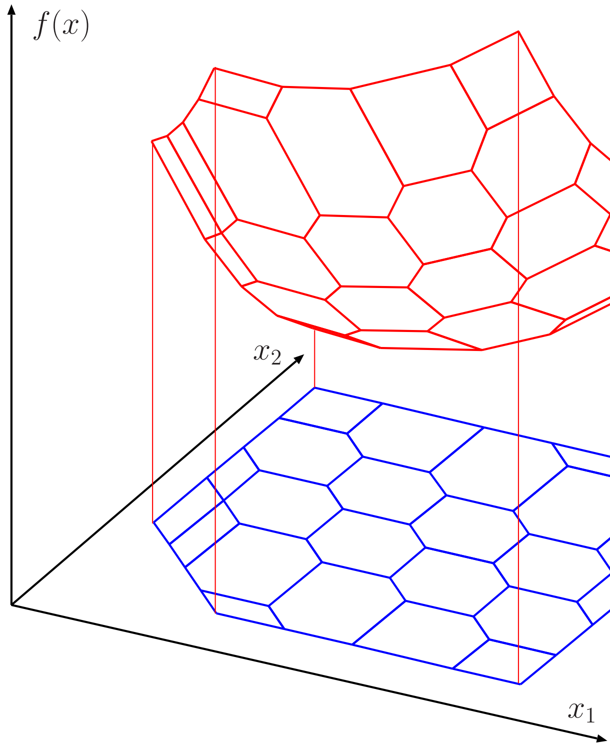
$f$ is convex-extensible

$\Leftrightarrow \exists$ convex $\overline{f}$:
$f(x) = \overline{f}(x)$

Convex-extensibility does not help much

# Discrete Convex Functions

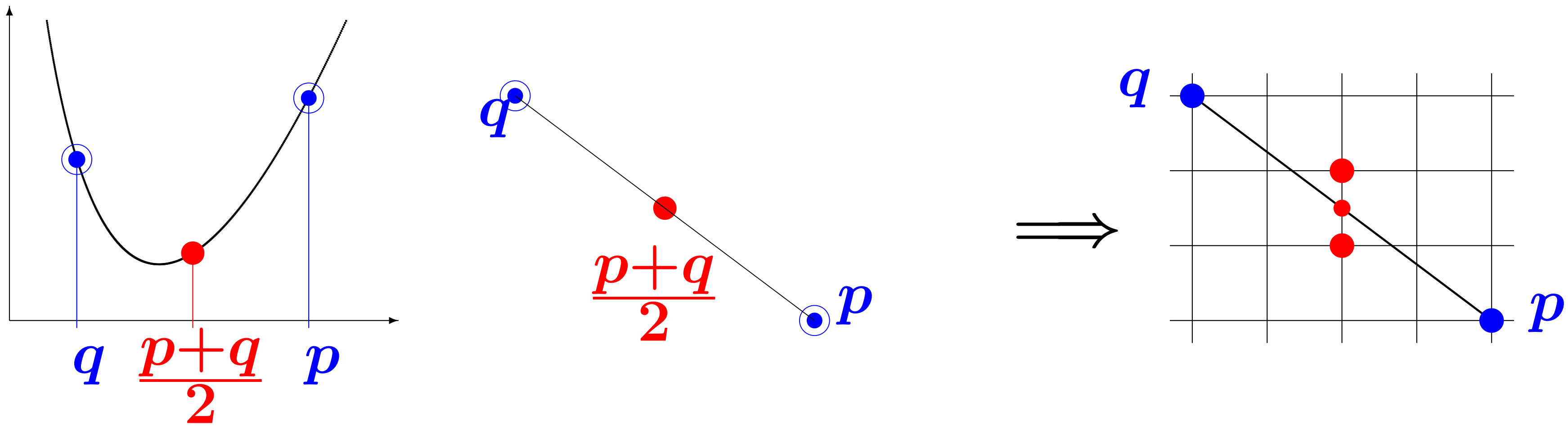



**L$^\natural$-convex fn** **M$^\natural$-convex fn**

# L♮-convexity from Mid-pt-convexity

(Murota 98, Fujishige–Murota 00)




Mid-point convex ($g : \mathrm{R}^n \to \mathrm{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

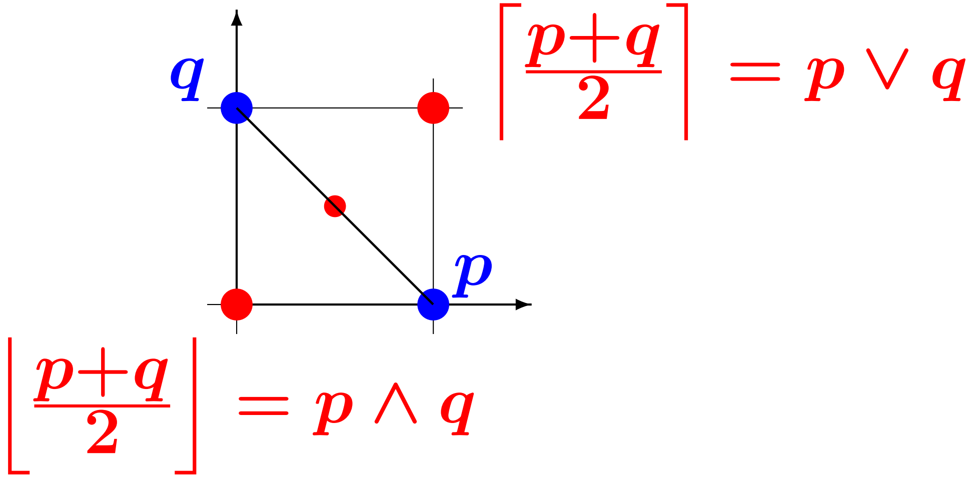
$\Longrightarrow$ Discrete mid-point convex ($g : \mathrm{Z}^n \to \mathrm{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$

L♮-convex function (L = Lattice)

# Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$




Discrete mid-pt convexity:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

$\iff$ Submodularity:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

# L$^\natural$-convexity from Submodularity

## —Original definition of L$^\natural$-convexity—

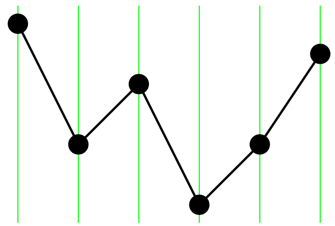
**Def:** $g : \mathbf{Z}^n \to \mathbf{R}$ is L$^\natural$-convex $\iff$

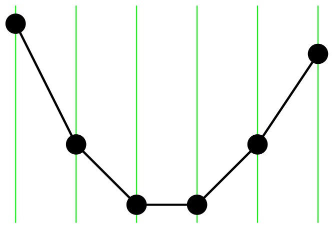
$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in $(p_0, p)$

$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1, 1)$

# Remark: L$^\natural$-convex vs Submodular

- L$^\natural$-convex function is convex-extensible
- General submodular function is NOT





**Fact 1:** Any $g : \mathbb{Z} \to \mathbb{R}$ is submodular

- Submodularity does not imply convexity

**Fact 2:** A function $g : \mathbb{Z} \to \mathbb{R}$ is L$^\natural$-convex

$\iff$ $g(p-1) + g(p+1) \geq 2g(p)$ for all $p \in \mathbb{Z}$

# L$^\natural$-convex Function: Examples

**Quadratic:** $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

**Separable convex:** For univariate convex $\psi_i$ and $\psi_{ij}$

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

**Range:** $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

**Submodular set function:** $\rho : 2^V \to \overline{\mathrm{R}}$

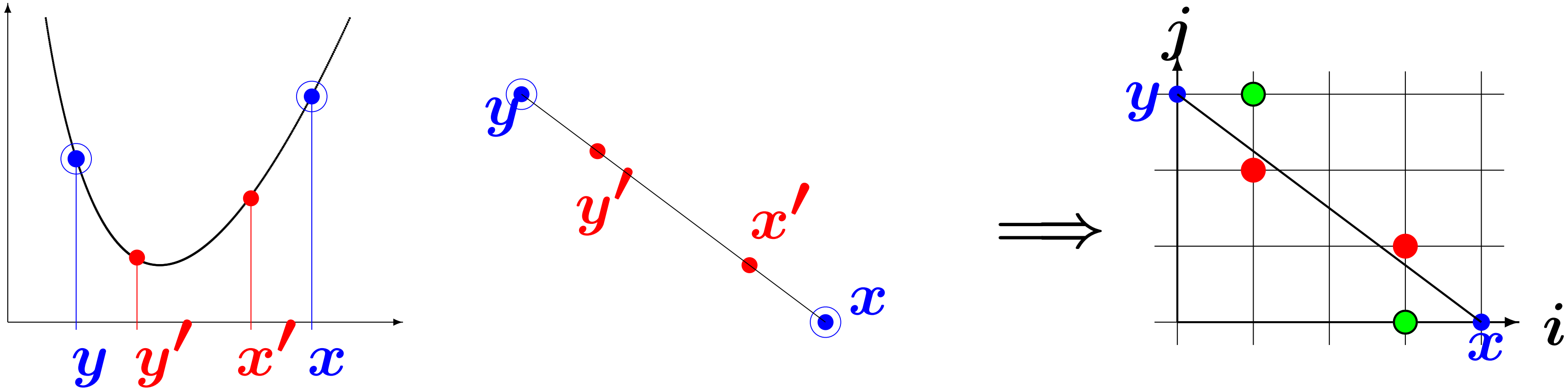
$\Leftrightarrow \rho(X) = g(\chi_X)$ for some L$^\natural$-convex $g$

**Multimodular:** $h : \mathrm{Z}^n \to \overline{\mathrm{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex $g$

# M♮-convexity from Equi-dist-convexity

(Murota 96, Murota–Shioura 99)




Equi-distance convex ($f : \mathrm{R}^n \to \mathrm{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y))$$

$\Longrightarrow$ **Exchange ($f : \mathrm{Z}^n \to \mathrm{R}$)** $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \Big[ f(x - e_i) + f(y + e_i),$$

$$\min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

**M♮-convex function** (M = Matroid)

# M$^\natural$-convex Function: Examples

**Quadratic:** $f(x) = \sum_i \sum_j a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow a_{ij} \geq 0, \quad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i, j\})$$

**Min value:** $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

**Matroid rank:** $f(X) = -$ rank of $X$

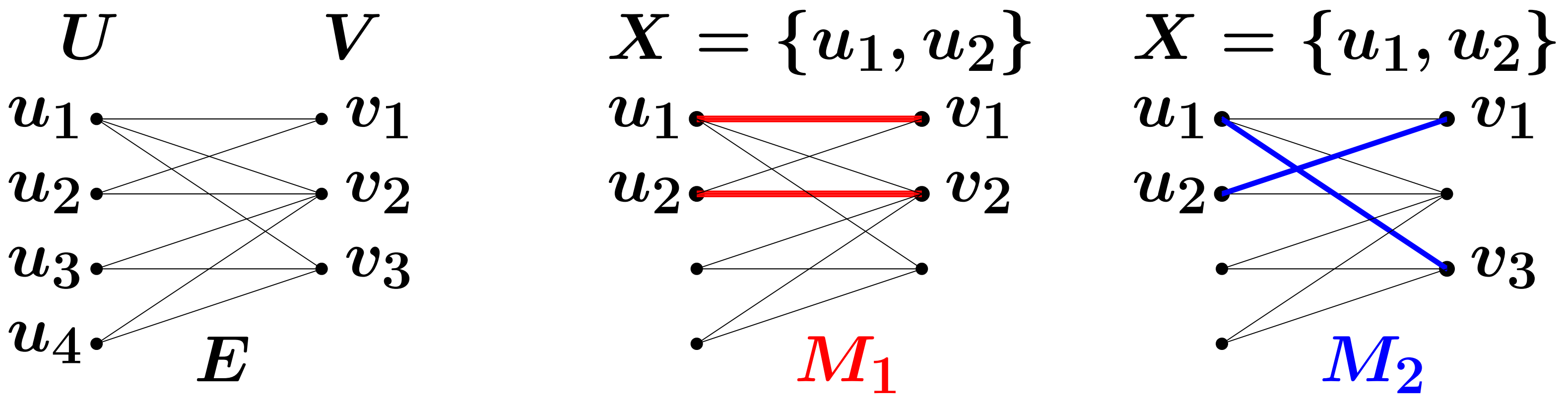
**Cardinality convex:** $f(X) = \varphi(|X|)$ ($\varphi$: convex)

**Separable convex:** $f(x) = \sum_i \varphi_i(x_i)$ ($\varphi_i$: convex)

**Laminar convex:** $f(x) = \sum_A \varphi_A(x(A))$ ($\varphi_A$: convex)

$\{A, B, ....\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

# Matching / Assignment




**Max weight for $X \subseteq U$** (**$w$: given weight**)

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

**Max-weight func $f$ is M$^\natural$-concave** (**Murota 96**)

- **Proof by augmenting path**
- **Extension to min-cost network flow**

# M$^\natural$-concavity = Gross Substitutes

Think of $f$ as a utility function

M$^\natural$-concave $\iff$ Gross substitutes $(+**)$

Reijnierse–van Gallekom–Potters 02, Fujishige-Yang 03
Danilov-Koshevoy-Lang 03, M.-Tamura 03

Gross substitutes: ($f$: utility, $p$: price)

$x \in \arg\max(f - p)$, $p \leq q$,

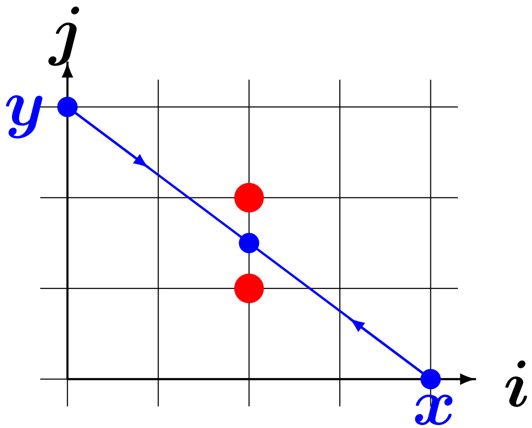
$\Rightarrow \quad \exists\, y \in \arg\max(f - q)$ : $y_i \geq x_i$ if $p_i = q_i$

$\Longrightarrow$ Applications to economics / game theory

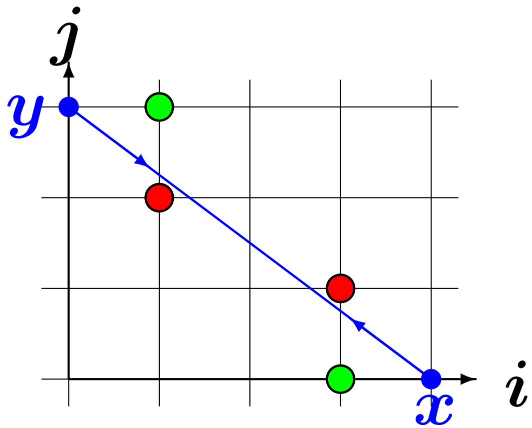
# Summary: Defs of L$^\natural$/M$^\natural$-convexity

| Continuous $\mathbb{R}^n \to \mathbb{R}$ | | Discrete $\mathbb{Z}^n \to \mathbb{R}$ |
|---|---|---|
| mid-pt convex | $\longrightarrow$ | disc mid-pt convex |
| $\Updownarrow$ | discr | (L$^\natural$-convex) |
| convex | | |
| $\Updownarrow$ | discr | (M$^\natural$-convex) |
| equi-dist convex | $\longrightarrow$ | exchange property |

L$^\natural$




M$^\natural$

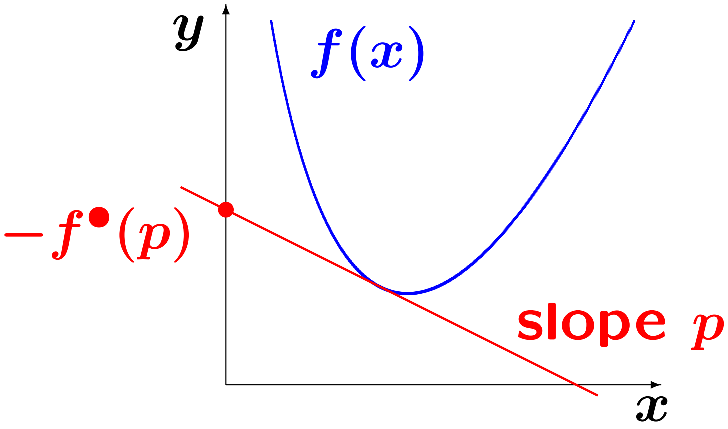


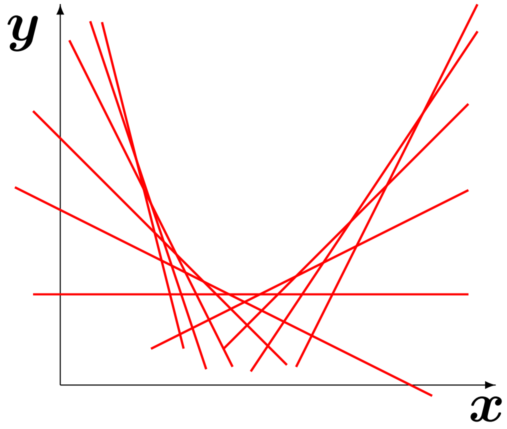
# B3.

## Conjugacy

## — Legendre transform

# Discrete Legendre Transform




$$f^{\bullet}(p) = \max_{x \in \mathbf{Z}^n} \{p \cdot x - f(x)\}$$

$\Rightarrow$ **If** $f : \mathrm{Z}^n \to \mathrm{Z}$, **then** $f^{\bullet} : \mathrm{Z}^n \to \mathrm{Z}$

**(integer-valued)**

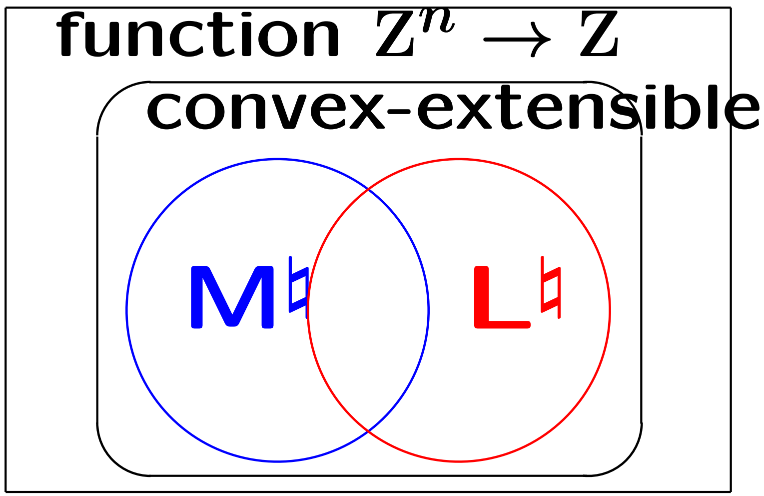
# M-L Conjugacy Theorem

**Integer-valued discrete fn** $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

**Legendre transform:** $f^\bullet(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

**M♮-convex and L♮-convex are conjugate**

$f \mapsto f^\bullet = g \mapsto g^\bullet = f$ **(Murota 98)**



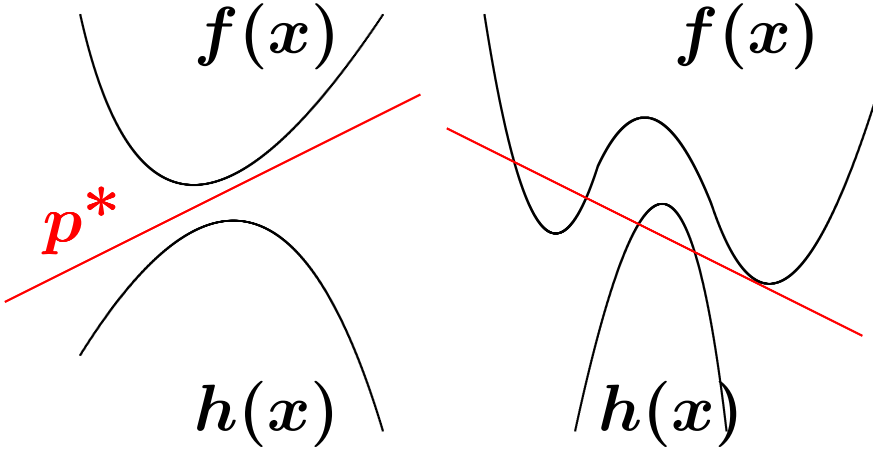
# B4.

## Duality

# Discrete Separation Theorem

$f : \mathbf{Z}^n \to \mathbf{R}$ "convex"

$h : \mathbf{Z}^n \to \mathbf{R}$ "concave"




- $f(x) \geq h(x) \quad (\forall x \in \mathbf{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbf{R}, \quad \exists\ p^* \in \mathbf{R}^n$:

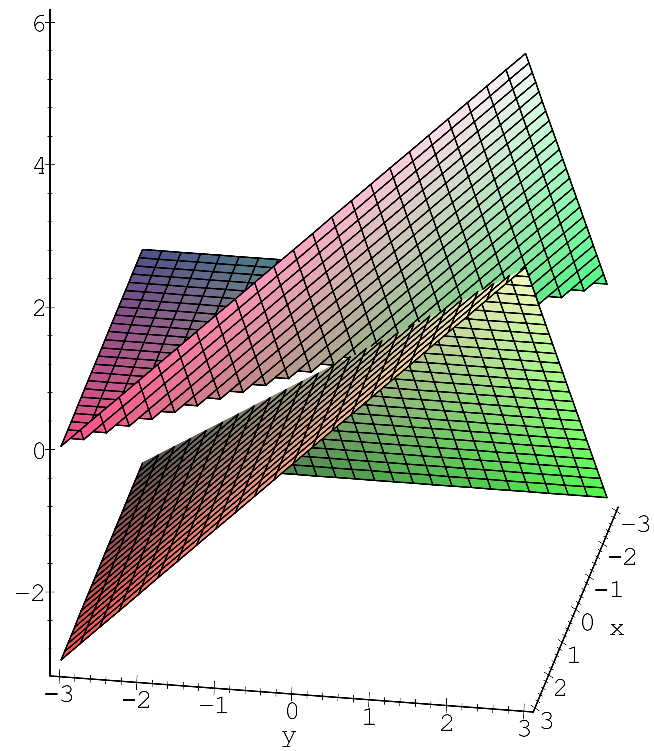
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbf{Z}^n)$$

- $f$, $h$: **integer-valued** $\Rightarrow \alpha^* \in \mathbf{Z}, \quad p^* \in \mathbf{Z}^n$

# Difficulty of Discrete Separation (1)

$f(x,y) = \max(0, x+y)$ **convex**

$h(x,y) = \min(x,y)$ **concave**




**nonintegral separation**

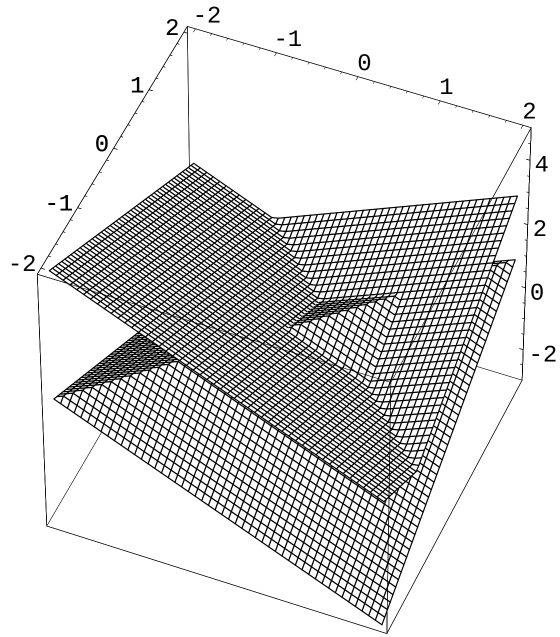
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

# Difficulty of Discrete Separation (2)

## Even real-separation is nontrivial

$f(x,y) = |x+y-1|$ **convex**

$h(x,y) = 1 - |x-y|$ **concave**




- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathbf{Z}^2)$ **true**
- **No** $\alpha^* \in \mathbf{R}$, $p^* \in \mathbf{R}^2$: $f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$

$\because$ $f = 0 < h = 1$ **at** $(x,y) = (1/2, 1/2)$

# Discrete Separation Theorems

(Murota 96/98)

## M-separation Thm (for M$^\natural$-convex)

$\Rightarrow$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn = M$^\natural$-convex fn)

## L-separation Thm (for L$^\natural$-convex)

$\Rightarrow$ Discrete separ. for submod. set fn (Frank 82)

(submod. set fn = L$^\natural$-convex fn on 0–1 vectors)

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(1) $f(x) \geq h(x) \quad (\forall x) \Rightarrow \; \exists\, \alpha^* \in \mathbf{R}, \; \exists\, p^* \in \mathbf{R}^n$:

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbf{Z}^n)$$

(2) $f$, $h$: integer-valued $\Rightarrow \alpha^* \in \mathbf{Z}, \; p^* \in \mathbf{Z}^n$

# Min-Max Duality

$f$: M$^\natural$-convex, $h$: M$^\natural$-concave ($\mathbf{Z}^n \to \mathbf{Z}$)

Legendre–Fenchel transform

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\}$$

**Fenchel-type Duality Thm** (Murota 96, 98)

$$\inf_{x\in\mathbf{Z}^n}\{f(x) - h(x)\} = \sup_{p\in\mathbf{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

self-conjugate ($f^\bullet$: L$^\natural$-convex, $h^\circ$: L$^\natural$-concave)

$\Longrightarrow$ Edmonds' matroid intersection thm

# Relation among Duality Thms

| Discrete Convex | Combinatorial Opt. |
|---|---|
| M-separation <br> $f(x) \geq \boxed{\text{Lin}} \geq h(x)$ | Fenchel duality (Fujishige 84) <br> matroid intersect. (Edmonds 70) |
| $\Updownarrow$ | $\Updownarrow$ |
| Fenchel duality <br> $\inf\{f - h\}$ <br> $= \sup\{h^{\circ} - f^{\bullet}\}$ | $\Rightarrow$ discrete separ. for submod (Frank 82) <br> $\Rightarrow$ valuated matroid intersect. (Murota 96) |
| $\Updownarrow$ | $\Downarrow$ |
| L-separation <br> $f^{\bullet}(p) \geq \boxed{\text{Lin}} \geq h^{\circ}(p)$ | weighted matroid intersect. (Edmonds 79, Iri-Tomizawa 76, Frank 81) |

# Summary (Part of "Basics")

- **Submodularity vs Convexity**
  submodular=convex (1980's), submodular=concave
- **Definitions of L$^\natural$-convex and M$^\natural$-convex**
  mid-point convexity, exchange, gross substitutes
- **Conjugacy**
  discrete Legendre transform, L$^\natural$ $\leftrightarrow$ M$^\natural$, $f^{\bullet\bullet} = f$
- **Duality**
  discrete separation, L-/M-separ., Fenchel min-max

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- Global minimality $\Longleftrightarrow$ local minimality
- Minimization algorithms: greedy, subm-min, scaling
- Continuous Variables
- Applications: OR, game, economics, matrices

# T1.
# Discrete DC Programming

—Use of Conjugacy

T. Maehara, K. Murota: A framework of discrete DC programming by discrete convex analysis, 2013

# Discrete DC Program

**DC = Difference of Convex**

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} \qquad g, h: \text{ "convex"}$$

**Convexity:** $\mathrm{M}^\natural$–$\mathrm{M}^\natural$, $\mathrm{M}^\natural$–$\mathrm{L}^\natural$, $\mathrm{L}^\natural$–$\mathrm{M}^\natural$, $\mathrm{L}^\natural$–$\mathrm{L}^\natural$

**Variable:** $x \in \mathbf{Z}^n$, $x \in \{0, 1\}^n$

# DC Algorithm (Pham Dinh Tao, 1985(ca.))

$$\min\{g(x) - h(x)\} \quad \Longrightarrow \quad \min\{g(x) - \langle p, x\rangle\}$$

**subgradient** $p \in \partial h(x)$

**Algorithm 1** DC algorithm

Let $x^{(1)}$ be an initial solution
**for** $k = 1, 2, \ldots$ **do**
(Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)})$
(Primal phase) Pick $x^{(k+1)} \in \text{argmin}\,(g - p^{(k)})$
**if** $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**
Return $x^{(k)}$
**end if**
**end for**

- **concave-convex proc (Yuille–Rangarajan 03)**
- **submod-supermod proc (Narasimhan–Bilmes 05)**

**cf. supermod-submod proc, mod-mod proc**
**(Iyer–Bilmes 12; Iyer–Jegelka–Bilmes 13)**

# Integral Subgradients & Biconjugacy

**In general:** $\partial_{\mathrm{Z}} f(0) = \emptyset, \qquad f^{\bullet\bullet} \neq f$

**Example:** $D = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}$

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2, & x \in D, \\ +\infty, & \text{o.w.} \end{cases}$$

$D$ is "convex": $\mathrm{conv}(D) \cap \mathrm{Z}^n = D$

$\partial f_{\mathrm{R}}(0) = \{(1/2, 1/2, 1/2)\}, \qquad \partial_{\mathrm{Z}} f(0) = \emptyset$

$$f^{\bullet\bullet}(0) = - \inf_{p \in \mathbb{Z}^3} \max\{0, |p_1 + p_2 - 1|, |p_2 + p_3 - 1|, |p_3 + p_1 - 1|\}$$

$f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$

# Subgradient of M♮-/L♮-convex Func

For $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

**integral subdifferential / subgradient**

$$\partial f(x) = \{p \in \mathbf{Z}^n \mid f(y) - f(x) \geq \langle p, y - x \rangle \quad (\forall y)\}$$

$f$: M♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: L♮-convex set

$$-p_i \leq f(x - \chi_i) - f(x), \qquad p_j \leq f(x + \chi_j) - f(x),$$

$$p_j - p_i \leq f(x - \chi_i + \chi_j) - f(x) \qquad (\forall i, j)$$

$f$: L♮-convex $\Rightarrow$ $\partial f(x) \neq \emptyset$: M♮-convex set

$$f(x) - f(x - \chi_A) \leq \langle p, \chi_A \rangle \leq f(x + \chi_A) - f(x) \quad (\forall A)$$

# Optimality Conditions $\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\}$

$x$: **global opt** $\iff$ $\partial_\epsilon g(x) \supseteq \partial_\epsilon h(x) \quad (\forall \epsilon \geq 0)$

$$\Downarrow$$

$$\boxed{\partial g(x) \supseteq \partial h(x)}$$

$$\Downarrow$$

$x$: **local opt, i.e.,** $x$: **minimum in**

$$U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$$

$$\partial_\epsilon f(x) = \{p \in \mathrm{R}^n \mid f(y) - f(x) \geq \langle p, y - x\rangle - \epsilon \quad (\forall y)\}$$

# Discrete DC Algorithm

$$\min_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} \quad \Longrightarrow \quad \min_{x \in \mathbf{Z}^n} \{g(x) - \langle p, x \rangle\}$$

**integral subgradient $p \in \partial h(x)$**

**Algorithm 2** Discrete DC algorithm

Let $x^{(1)}$ be an initial solution
**for** $k = 1, 2, \ldots$ **do**
  (Dual phase) Pick $p^{(k)} \in \partial h(x^{(k)}) \setminus \partial g(x^{(k)})$
  (Primal phase) Pick $x^{(k+1)} \in \mathrm{argmin}\,(g - p^{(k)})$
  **if** $(g - p^{(k)})(x^{(k)}) = (g - p^{(k)})(x^{(k+1)})$ **then**
    Return $x^{(k)}$
  **end if**
**end for**

- **monotone decreasing $g(x) - h(x)$**
- **$\partial g(x) \supseteq \partial h(x)$ guaranteed**
- **local optimality within $U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$**

# Testing for Local Opt: $\partial g(x) \supseteq \partial h(x)$

M$^\natural$–M$^\natural$: poly-time (explicit $O(n^2)$ ineqs)

M$^\natural$–L$^\natural$: poly-time ($O(n^2)$ M-min)

L$^\natural$–M$^\natural$: poly-time (L-min, submod-min)

L$^\natural$–L$^\natural$: NP-hard (submod containment)

$\Rightarrow$ M$^\natural$–M$^\natural$ by Toland-Singer duality

# Discrete Toland-Singer Duality

$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$

$g : \mathbf{Z}^n \to \mathbf{Z}$: any fn;    $h : \mathbf{Z}^n \to \mathbf{Z}$: M$^\natural$- or L$^\natural$-convex

**Toland-Singer Duality Thm** (Maehara-Murota 13)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^\bullet(p) - g^\bullet(p)\}$$

(Proof) Integral biconjugacy: $h^{\bullet\bullet} = h$.

$$\begin{aligned}
&\inf_x \{g(x) - h(x)\} = \inf_x \{g(x) - h^{\bullet\bullet}(x)\} \\
&= \inf_x \{g(x) - \sup_p \{\langle p, x\rangle - h^\bullet(p)\}\} \\
&= \inf_x \inf_p \{g(x) - \langle p, x\rangle + h^\bullet(p)\} \\
&= \inf_p \{h^\bullet(p) - \sup_x \{\langle p, x\rangle - g(x)\}\} = \inf_p \{h^\bullet(p) - g^\bullet(p)\}.
\end{aligned}$$

# Summary (“DC Programming”)

- **Theoretical framework**
  M$^\natural$–M$^\natural$, L$^\natural$–M$^\natural$ (subm), M$^\natural$–L$^\natural$ (superm), L$^\natural$–L$^\natural$,
  integral subgradient, biconjugacy, Toland-Singer
- **Local optimality condition**
  $\partial g(x) \supseteq \partial h(x)$, checking this condition
- **Algorithm**
  monotone, finite, local opt
- **Using discrete convex analysis for nonconvex prob**

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- DC Representability
- Hardness of minimization
- Approximation guarantee (for some 01 cases)
- Computational results (will come)

# T2.
# Valuated Matroid-Based Submodular Welfare Algorithm

—Use of Duality

T. Maehara, K. Murota: Valuated matroid-based algorithm for submodular welfare problem, 2013

# Submodular Welfare Maximization

Maximize $f_1(X_1) + \cdots + f_m(X_m)$

s.t. $(X_1, \ldots, X_m)$: partition of $V$

$\Rightarrow$ NP-hard for general submodular $f_i$

$f_i$: M$^\natural$-concave (matroid valuation)

$\Rightarrow$ poly-time solvable by

valuated matroid intersection/
partition algorithm (Murota 96)

$\Rightarrow$ Add "neg-cycle" heuristics for general $f_i$

cf. ejection chain (Glover 96)

# Budgeted Allocation Problem

(Garg-Kumar–Pandit 01)

$$f_i(X) = \min\{\sum_{x \in X} b_{ix},\ B_i\} \quad (X \subseteq V)$$

Maximize $f_1(X_1) + \cdots + f_m(X_m)$

s.t. $X_1, \ldots, X_m$: subpartition of $V$

$m$ bidders, $n$ items $V$

bidder $i$ has budget $B_i$

will pay $b_{ix} > 0$ for item $x$

# Budgeted Allocation Problem (R102570)

$m = 10,\ n = 25,\ B_i = 70$

LLN = Lehmann-Lehmann-Nisan (06)

| Opt | LLN (ratio) | Proposed (ratio) | Cycle |
|---|---|---|---|
| 646 | 633 (0.979) | 643 (0.995) | 0/5 |
| 649 | 635 (0.978) | 646 (0.995) | 0/1 |
| 650 | 640 (0.984) | 650 (1.000) | 0/2 |
| 639 | 625 (0.978) | 636 (0.995) | 0/3 |
| 648 | 631 (0.973) | 648 (1.000) | 0/2 |
| 649 | 639 (0.984) | 649 (1.000) | 0/2 |
| 648 | 635 (0.979) | 648 (1.000) | 0/2 |
| 646 | 636 (0.984) | 645 (0.998) | 0/1 |
| 649 | 644 (0.992) | 649 (1.000) | 0/1 |
| 643 | 631 (0.981) | 642 (0.998) | 0/1 |
| Aver. | (0.981) | (0.998) | |

"Cycle": contractions / detected negative cycles

# Min-GAP (Generalized Assignment Problem)

$$f_i(X) = \sum_{x \in X} c_{ix} \quad \text{if} \quad \sum_{x \in X} b_{ix} \leq B_i$$

Minimize $f_1(X_1) + \cdots + f_m(X_m)$

s.t. $X_1, \ldots, X_m$: partition of $V$

$m$ bidders, $n$ items $V$

bidder $i$ has budget $B_i$

will pay $b_{ix} > 0$ for item $x$

gets profit $c_{ix} 0$ for item $x$

# Min-GAP (ORLIB: types a, b)

**Generalized Assignment Problem** Chu–Beasley (1997)

| Problem | Best kwn | Proposed (ratio) | Cycle | |
|---|---|---|---|---|
| a05100 | 1698 opt | 1698 (1.000) | 0/0 | |
| a05200 | 3235 opt | 3235 (1.000) | 0/0 | |
| a10100 | 1360 opt | 1360 (1.000) | 0/0 | |
| a10200 | 2623 opt | 2623 (1.000) | 0/0 | |
| a20100 | 1158 opt | 1158 (1.000) | 0/0 | |
| a20200 | 2339 opt | 2341 (1.001) | 0/0 | |
| b05100 | 1843 | 1855 (1.006) | 113/230 | P |
| b05200 | 3552 | 3565 (1.003) | 4968/5197 | P |
| b10100 | 1407 | 1413 (1.004) | 0/7 | |
| b10200 | 2828 | 2848 (1.007) | 1764/1918 | P |
| b20100 | 1166 | 1168 (1.002) | 0/3 | |
| b20200 | 2340 | 2343 (1.001) | 0/9 | |

"Cycle": contractions / detected negative cycles

P: feasible point difficult to find

# Min-GAP (ORLIB: types c, d)

**Generalized Assignment Problem** Chu–Beasley (1997)

| Problem | Best kwn | | Proposed (ratio) | Cycle | |
|---|---|---|---|---|---|
| c05100 | 1931 | opt | 1969 (1.019) | 63/167 | P |
| c05200 | 3456 | opt | 3482 (1.007) | 4261/4509 | P |
| c10100 | 1402 | opt | 1415 (1.009) | 531/567 | P |
| c10200 | 2806 | opt | 2820 (1.004) | 4863/5011 | P |
| c20100 | 1243 | opt | 1257 (1.011) | 247/226 | P |
| c20200 | 2391 | | 2402 (1.004) | 1241/1311 | PF |
| d05100 | 6353 | opt | 6729 (1.059) | 9/264 | P |
| d05200 | 12743 | | 13272 (1.041) | 2759/3337 | P |
| d10100 | 6349 | | 6662 (1.049) | 683/849 | PF |
| d10200 | 12436 | | 13058 (1.050) | 2131/2631 | P |
| d20100 | 6196 | | 6528 (1.053) | 645/699 | |
| d20200 | 12264 | | 12974 (1.057) | 2167/2473 | |

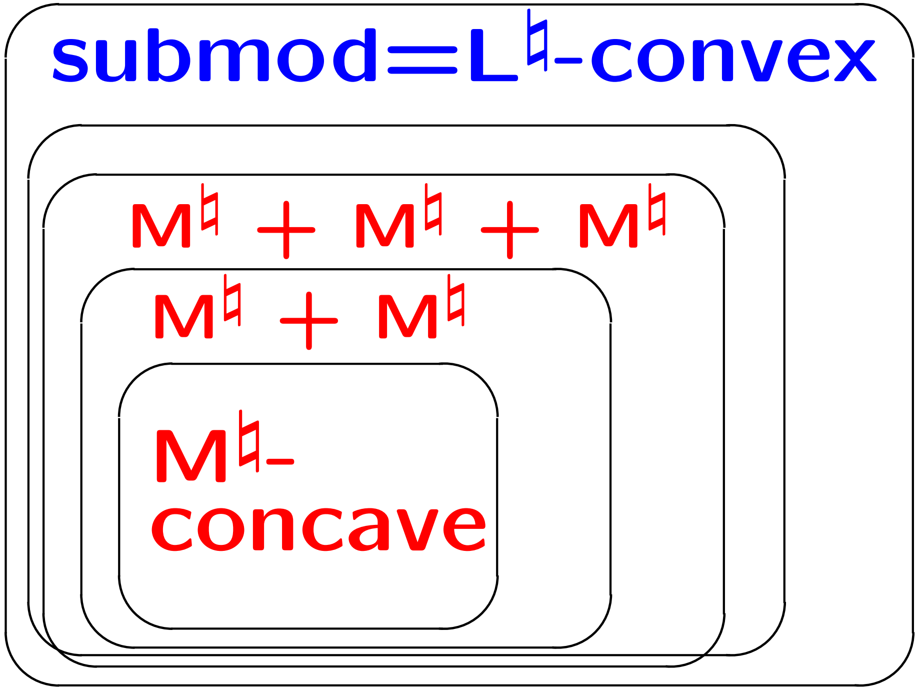
"Cycle": contractions / detected negative cycles

P: feasible point difficult to find; F: $G \neq G(M)$

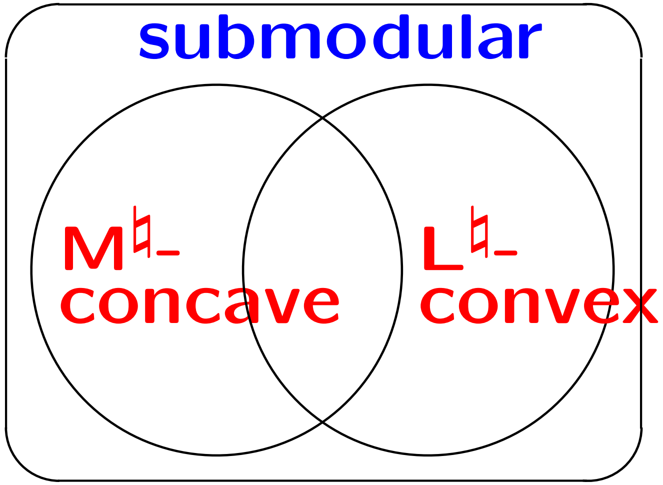
# Submodular Set Function in DCA

- **Submodular** set func = L$^\natural$-convex on $\{0,1\}^n$
- (Sums of) M$^\natural$-concave form a nice subclass

$f: \{0,1\}^n \to \mathrm{R}$




$f: \mathrm{Z}^n \to \mathrm{R}$



# M$\natural$-concave Set Functions

**Set function** $f : 2^V \to \mathbf{R} \cup \{-\infty\}$

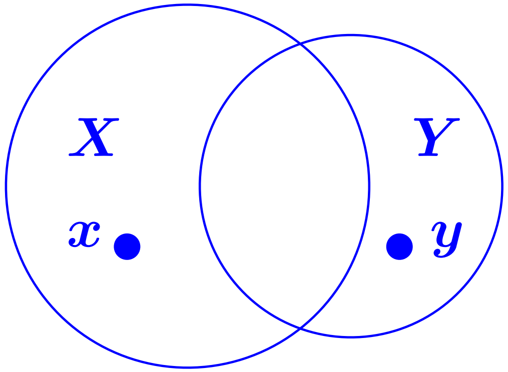
**M$\natural$-concave or matroid valuation**

$\iff \forall\ X, Y \subseteq V, \quad \forall\ x \in X \setminus Y$:

$$f(X) + f(Y)$$

$$\leq \max\{f(X - x) + f(Y + x),$$

$$\max_{y \in Y \setminus X} [f(X - x + y) + f(Y + x - y)]\}$$




**Dress–Wenzel (90,92) on bases**

# M$^\natural$-concave Set Functions

M$^\natural$-concave is submodular (NOT conversely)

M$^\natural$-concave forms a nice subclass for maximization

- $f(X) = \varphi(|X|)$ ($\varphi$: concave)
- $f(X) = \sum_{A \in \mathcal{T}} \varphi_A(|A \cap X|)$ ($\varphi_A$: concave)

  $\mathcal{T}$: laminar ($A, B \in \mathcal{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$)
- max-value $f(X) = \max\{a_i \mid i \in X\}$
- matroid rank (Fujishige 05)

  $f(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$
- weighted matroid rank ($w \geq 0$) (Shioura 09)

  $f(X) = \max\{w(I) \mid I : \text{independent}, \ I \subseteq X\}$

# Valuated Matroid Partition

$f_i$: matroid valuations ($2^V \to \overline{\mathrm{R}}$)

Maximize $f_1(X_1) + \cdots + f_m(X_m) =: \Phi(X)$
s.t. $\{X_1, \ldots, X_m\}$: partition of $V$

## Existence of (integral) subgradient (Murota 96)

(1) $(X_1^*, \ldots, X_m^*)$: optimal

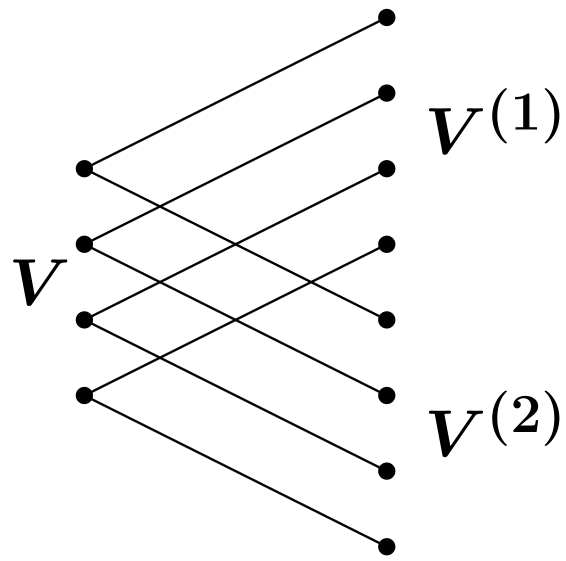
$\iff \exists\, p$ (certificate of optimality)

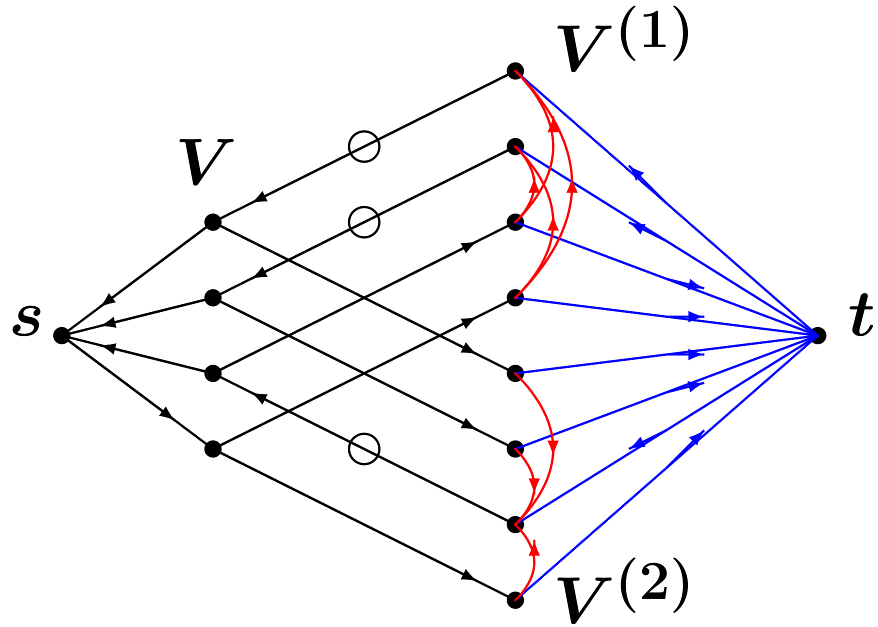
- $X_i^*$ maximizes $\boxed{f_i(X) - p(X)}$ $(\forall i)$
- $p(V) = 0$

(2) $f_i$ are integer-valued $\Rightarrow$ integral $p$

# Valuated Matroid Partition Algorithms

Extension of weighted matroid partition




Exchange: $f_i(X) - f_i(X - v + u)$

Sink arc: $f_i(X) - f_i(X \pm u)$

Valuated matroid intersection (Murota 96)

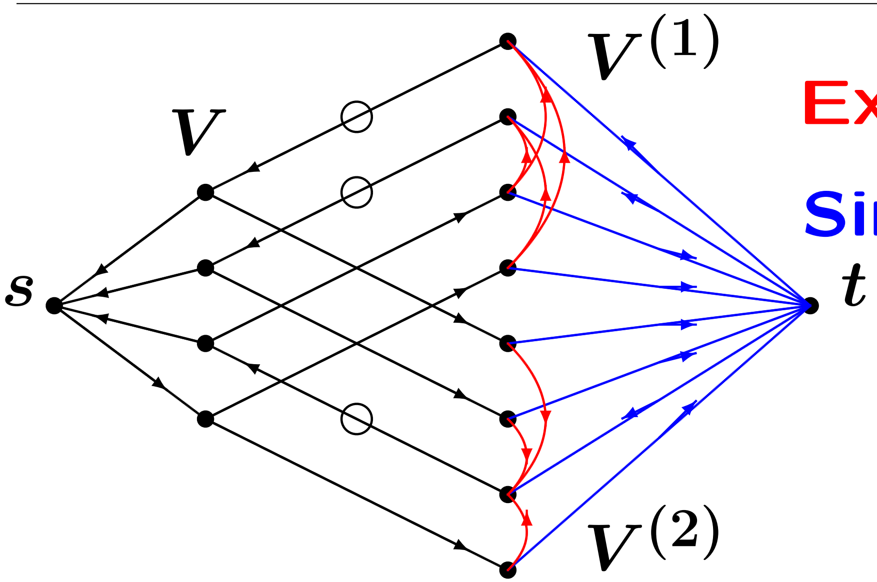
- cycle-canceling alg
- augmenting path alg
- "upper-bound lemma"
- "unique-max lemma"

# Augmenting Path Algorithm

**Maximize** $f_1(X_1) + \cdots + f_m(X_m) =: \Phi(X)$
**s.t.** $\{X_1, \ldots, X_m\}$: **partition of** $V$

**Algorithm 3** Augmenting path algorithm

1: Let $M = \emptyset$ be an initial solution
2: **for** $k = 1, \ldots, n$ **do**
3: Find a shortest path $P$ from source-vertex $s$ to sink-vertex $t$
4: Update $M \leftarrow (M \Delta P) \cap E$
5: **end for**




**Exchange:** $f_i(X) - f_i(X - v + u)$

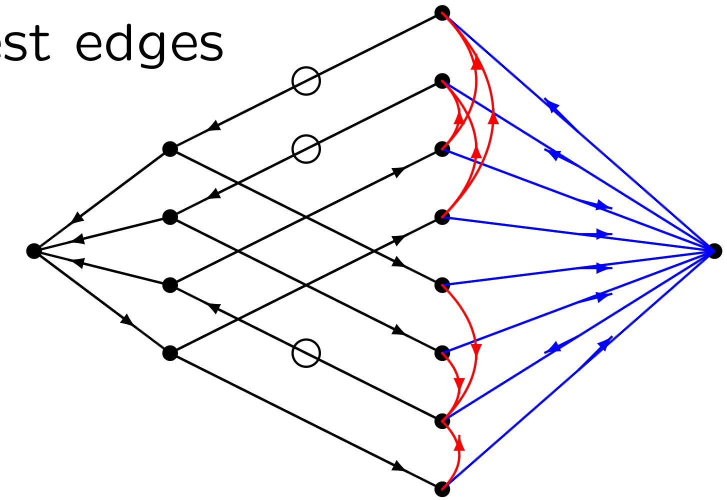
**Sink arc:** $f_i(X) - f_i(X \pm u)$

# Cycle-canceling Algorithm

**Maximize** $f_1(X_1) + \cdots + f_m(X_m) =: \Phi(X)$
**s.t.** $\{X_1, \ldots, X_m\}$: **partition of** $V$
$|X_1 \cup \cdots \cup X_m| = k$

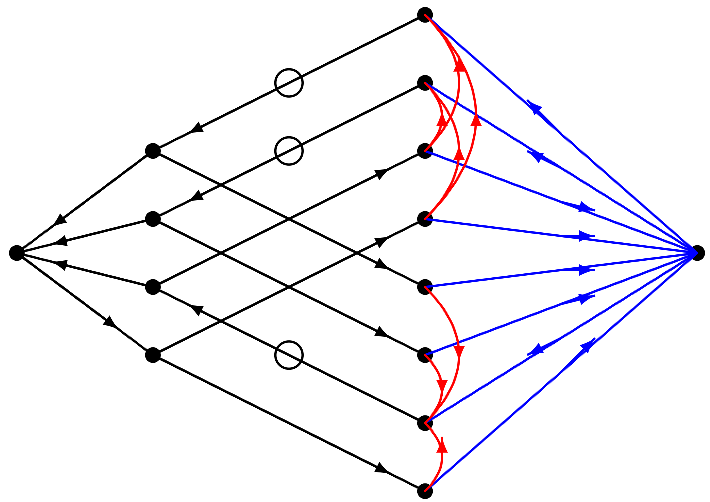
**Algorithm 4** Cycle-canceling algorithm

Let $M$ be an initial matching of size $k$
**loop**
    Find a negative cycle $C$ with the fewest edges
    **if** there are no negative cycles **then**
        **return** $M$ as an optimal solution
    **else**
        $M \leftarrow (M \Delta C) \cap E$
    **end if**
**end loop**



**Algorithm 5** Proposed algorithm for submodular welfare problem

Let $M = \emptyset$ be an initial solution<br>
**for** $k = 1, \ldots, n+1$ **do**<br>
  $G \leftarrow G(M)$<br>
  **loop**<br>
    Find a negative cycle $C$ with the fewest edges in $G$<br>
    **if** there are no negative cycles **then break end if**<br>
    Let $M' := (M \Delta C) \cap E$<br>
    **if** $\Phi(M') > \Phi(M)$ **then**<br>
      $M \leftarrow M'$;  $G \leftarrow G(M)$<br>
    **else**<br>
      Set $\gamma(e) \leftarrow 0$ for all $e \in C$ in $G$<br>
    **end if**<br>
  **end loop**<br>
  **if** $k < n$ **then**<br>
    Find a shortest path $P$ of $G$ from source $s$ to sink $t$<br>
    Update $M \leftarrow (M \Delta P) \cap E$<br>
  **end if**<br>
**end for**



# Summary (“Submodular Welfare”)

- **Algorithm**

  based on valuated-matroid partition algs

  exact for a subclass of submodular welfare

  negative-cycle heuristics

- **Computational results**

  fairly good performance

- **Using discrete convex analysis for nonconvex prob**

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- 1/2 approx ratio under an a posteriori assumption
- More computational experiments (will come)

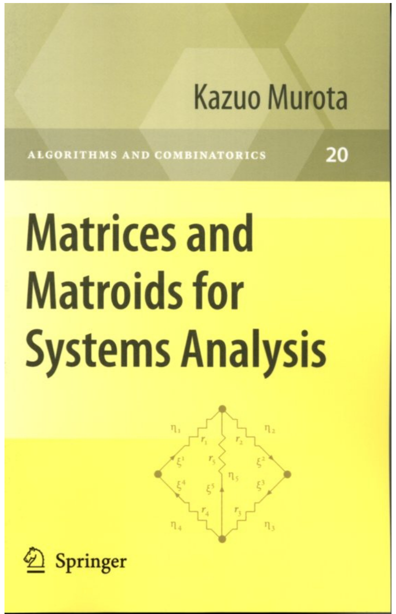
# Books

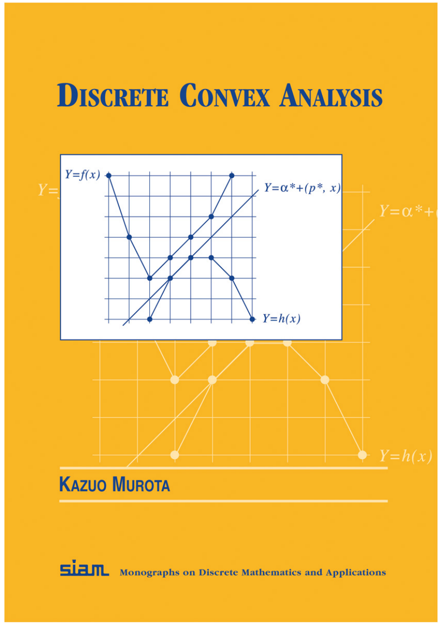
Murota: Matrices and Matroids for Systems Analysis, Springer, 2000/2010 (Chap.5)

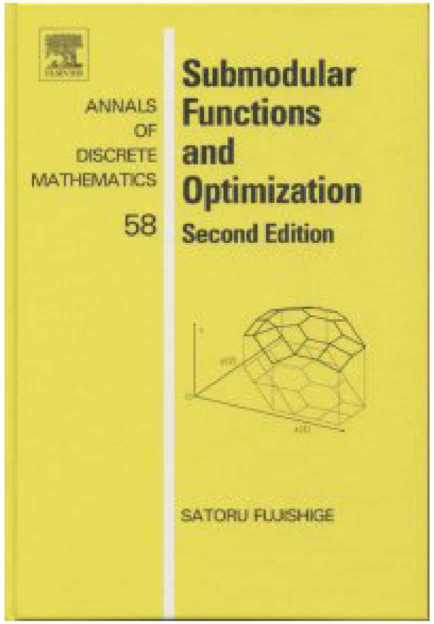
(valuated matroid intersection algorithm)

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)



# Survey/Slide/Video/Software

[Survey]
Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

[Slide] [Video]

http://www.misojiro.t.u-tokyo.ac.jp/ murota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)
http://www.misojiro.t.u-tokyo.ac.jp/DCP/

E N D