Kyoto U., Inst. Economic Research 2014-03-20

Bifurcation Theory for Hexagonal Agglomeration in Economic Geography

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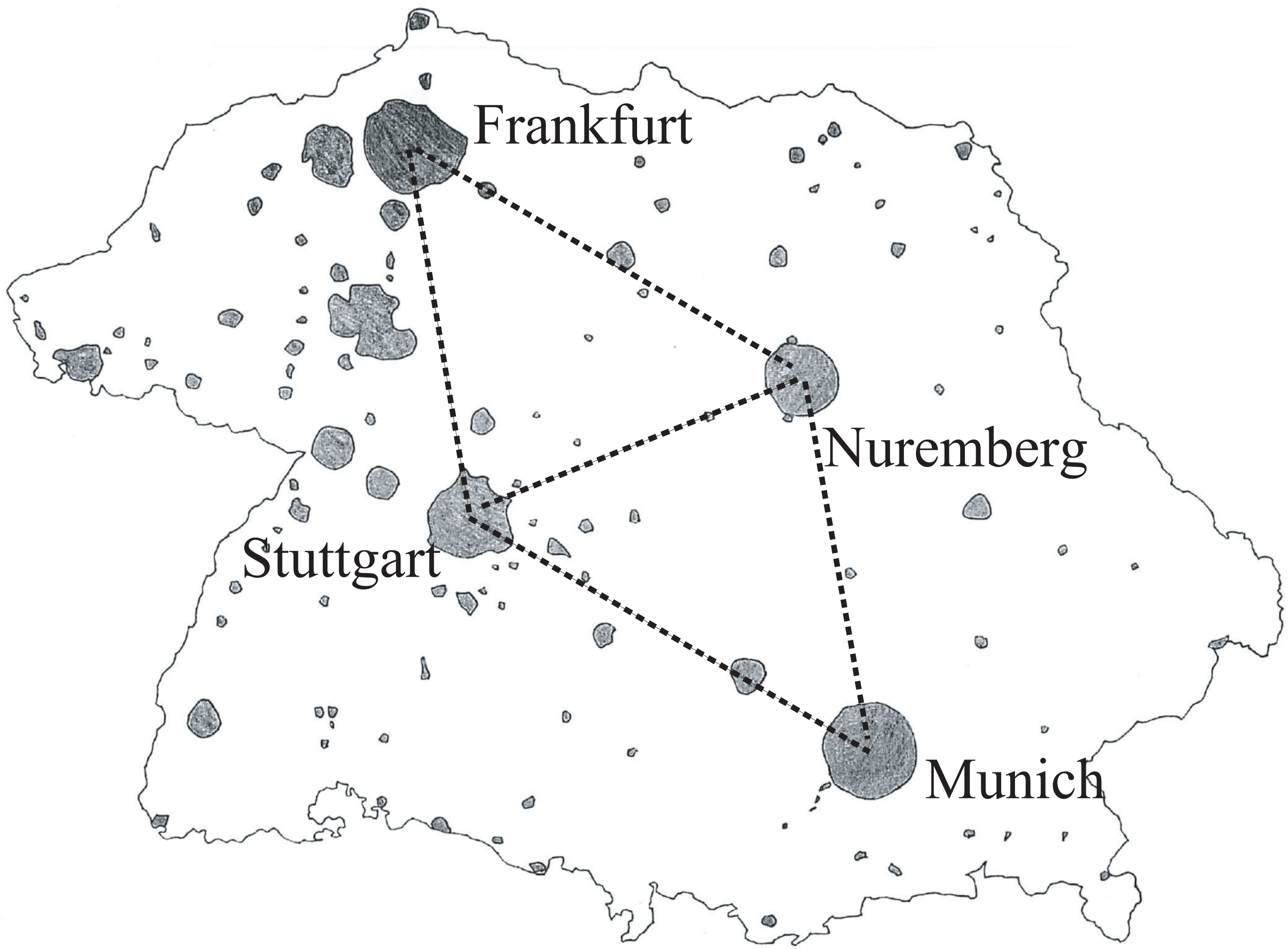
Joint work with

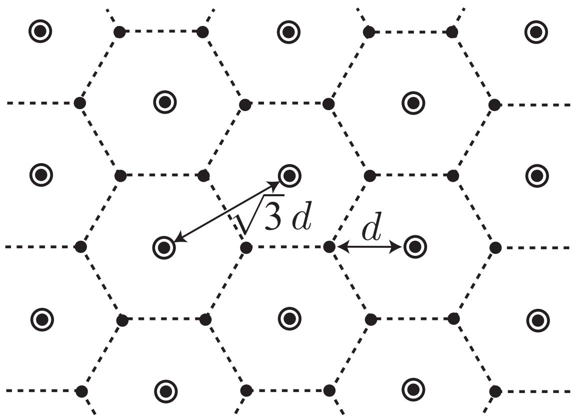
Kiyohiro Ikeda (池田清宏)

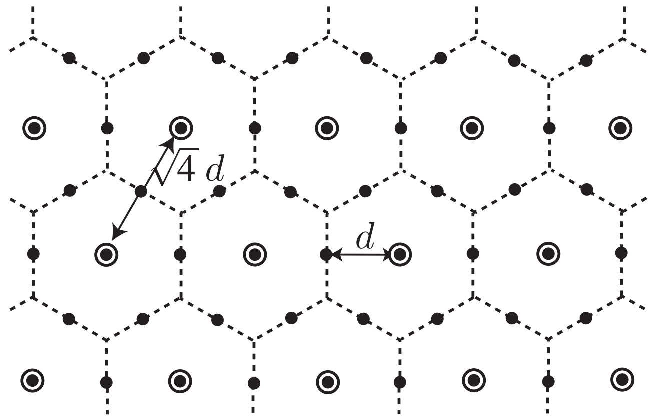
Southern Germany

Christaller 1933

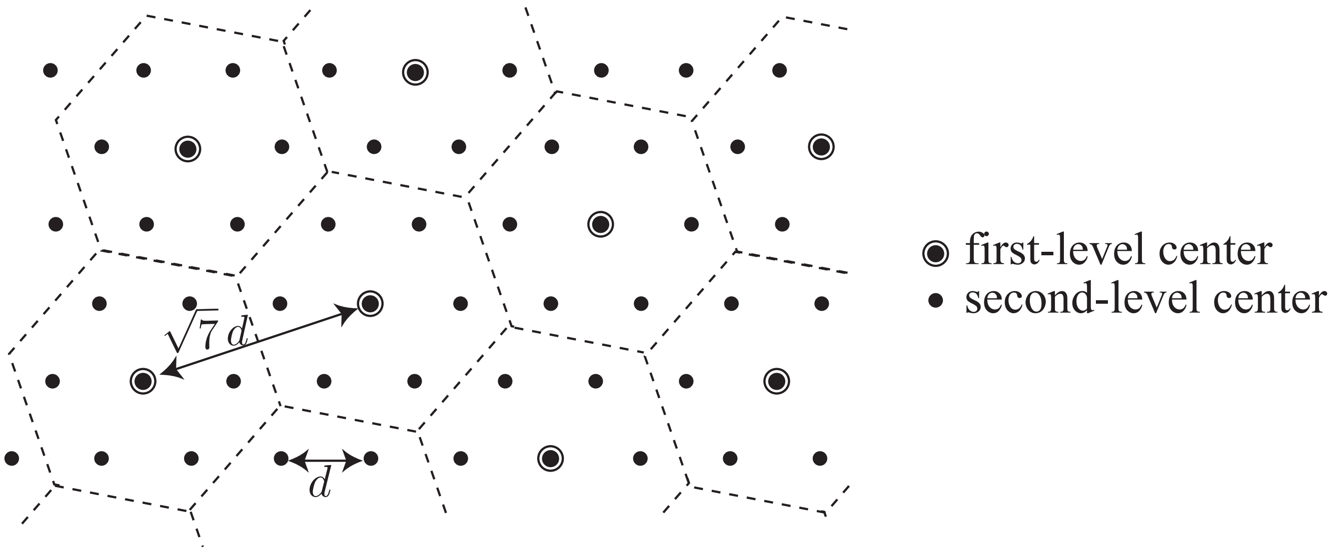


Christaller's Systems (Central Place Theory)





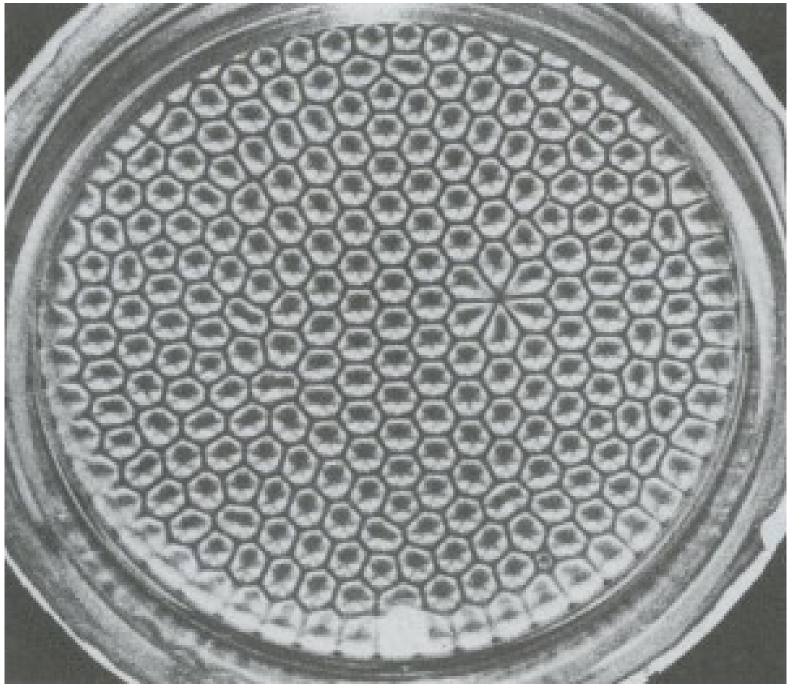
$\boldsymbol{k = 3}$ system (market principle) $\boldsymbol{k = 4}$ system (traffic principle)



$\boldsymbol{k = 7}$ system (administrative principle)

Bénard Convection in Fluid Dynamics

Hexagonal tessellation



Successful math analysis by

group-theoretic
bifurcation theory
(群論的分岐理論)

Koschmieder (1974) Benard convection, Adv in Chemical Physics

Bifurcation Theory for Hexagonal Agglomeration in Economic Geography

(分岐理論)
(人口集積)
(経済地理学)

Part 1: (background)

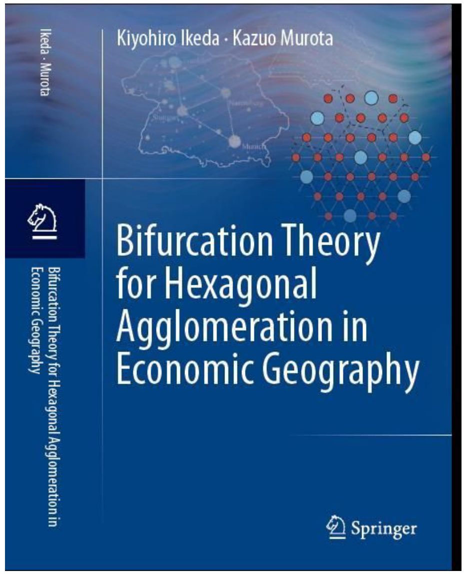
Economic Geography

Part 2: (result)

Hexagonal Agglomeration

Part 3: (methodology)

Group-Theoretic Bifurcation Theory



Part 1.

Economic Geography

Economic Geography (経済地理学)

von Thünen (1826): von Thünen Ring

Christaller (1933), Lösch (1940):
Central Place Theory (中心地理論)

New Economic Geography (新経済地理学)

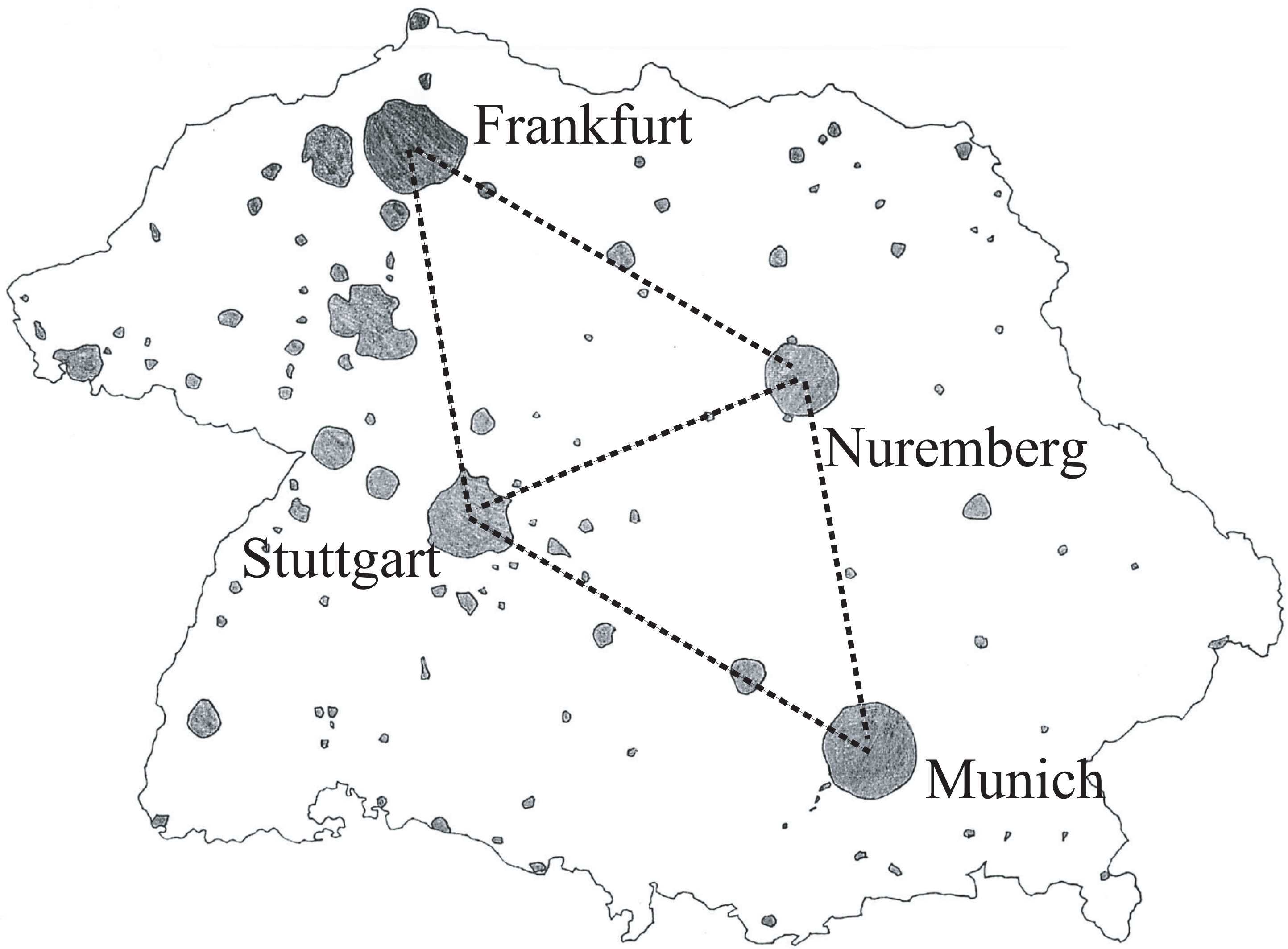
Krugman (1991):
Increasing returns and economic geography

Fujita, Krugman, Venables (1999):
The Spatial Economy:
Cities, Regions, and International Trade
(空間経済学：都市・地域・国際貿易の新しい分析)

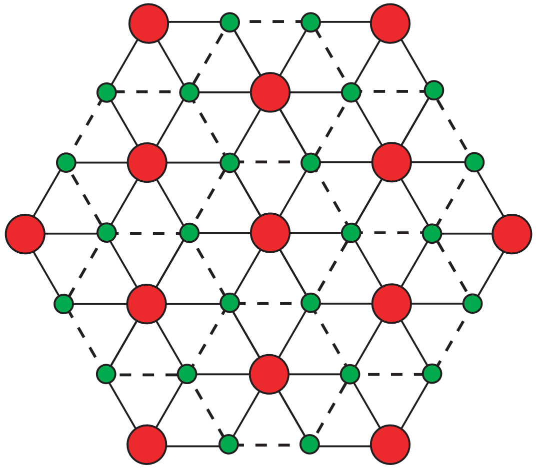
Fujita (2010): The evolution of spatial economics:
from Thünen to the New Economic Geography

Southern Germany

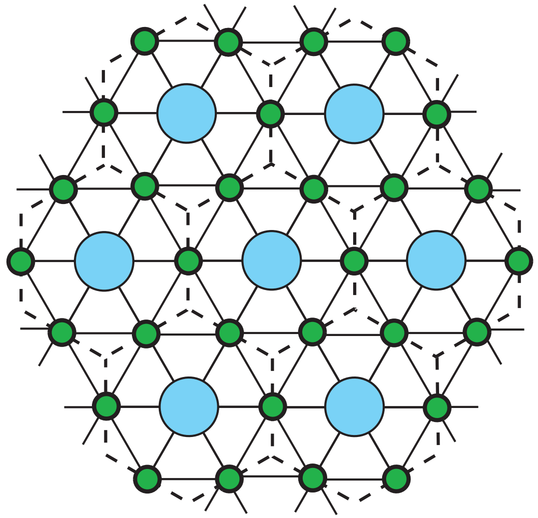
Christaller 1933



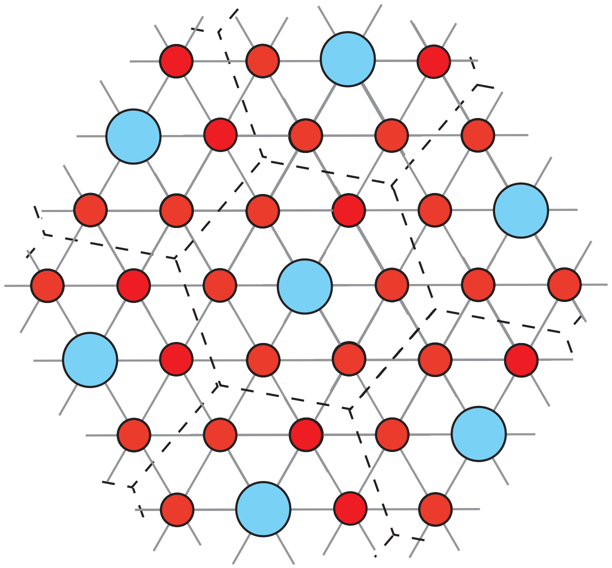
Christaller's Systems (Central Place Theory)



Christaller's $\boldsymbol{k = 3}$ system

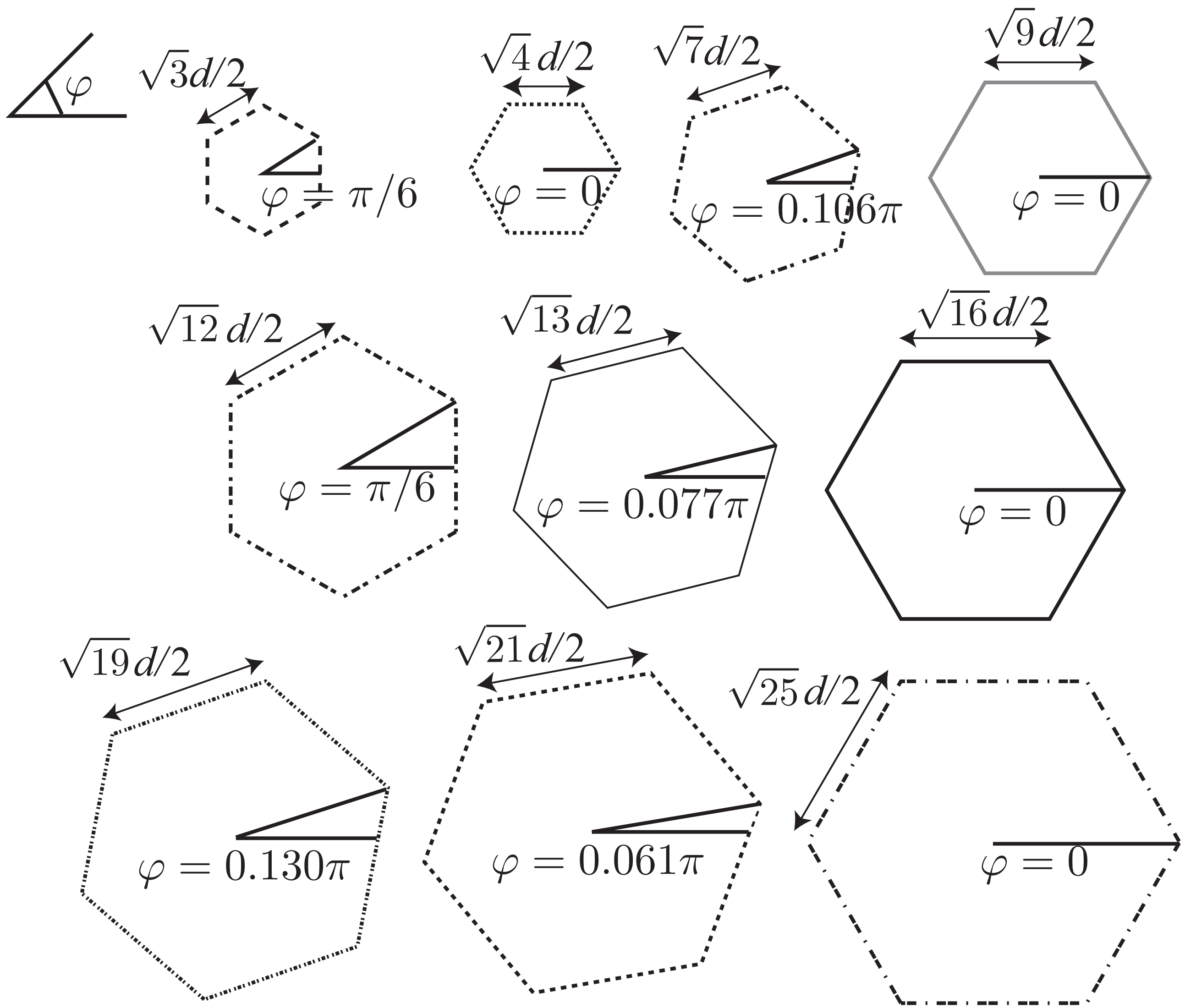


Christaller's $\boldsymbol{k = 4}$ system



Christaller's $\boldsymbol{k = 7}$ system

Lösch's Hexagons



Economic Geography / Central Place Theory

- descriptive / normative approach
- no mechanism (micro-economic, mathematical)

New Econ. Geography / Spatial Economics

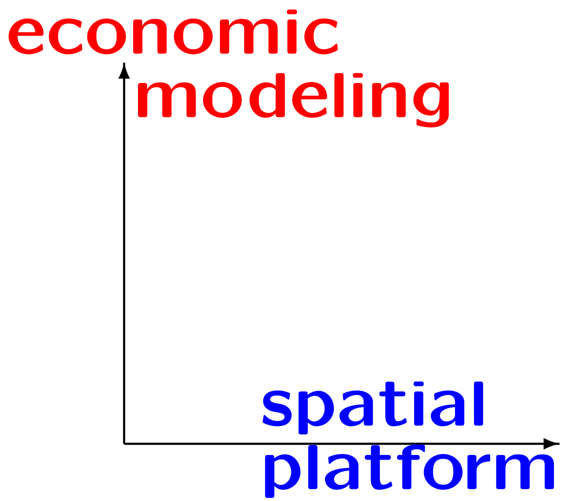
- micro-economic mechanism

  core-periphery model: transport cost,
  market equilibrium, population migration

Our Study

- mathematical mechanism

  pattern formation, bifurcation



Core–Periphery Model

An economy with n places: $i = 1, \ldots, n$

Two industrial sectors:

- agriculture: perfectly competitive, $\cdots$

- manufacturing: imperfectly competitive,

transport cost, increasing returns, $\cdots$

Two types of labour:

- farmers: immobile

- workers: mobile

....

....

Core-Periphery Model

- **Market equilibrium (short-run)**

  Given: λ_i: population in place i $(= 1, 2, \ldots, n)$

  τ: transport cost parameter

- **Population migration (long-run)**

$$\frac{d\lambda_i}{dt} = \boxed{F_i(\lambda, \tau)} \qquad i = 1, \ldots, n$$

  e.g.: Replicator dynamics (Krugman, 1991)

$$\boxed{F_i(\lambda, \tau)} = (\omega_i(\lambda, \tau) - \overline{\omega}(\lambda, \tau))\lambda_i, \qquad i = 1, \ldots, n$$

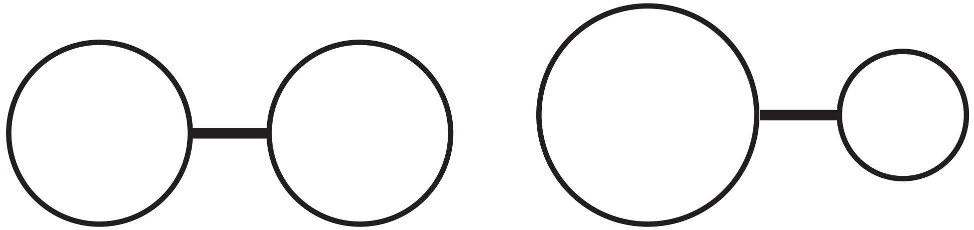
  – Market equil. $\Longrightarrow$ real wage $\omega_i = \omega_i(\lambda, \tau)$

  – Average real wage $\overline{\omega} = \sum_{i=1}^{n} \lambda_i \omega_i$

Two-Place Economy

$\lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \geq 0$

transport cost:
$T = 1/(1-\tau)$



(a) $\lambda_1 = \lambda_2$ (b) $\lambda_1 \neq \lambda_2$

$$Y_1 = \mu\lambda_1 w_1 + \frac{1-\mu}{2}, \qquad Y_2 = \mu\lambda_2 w_2 + \frac{1-\mu}{2}$$

$$G_1 = [\lambda_1 {w_1}^{1-\sigma} + \lambda_2 (w_2 T)^{1-\sigma}]^{\frac{1}{1-\sigma}}$$

$$G_2 = [\lambda_1 (w_1 T)^{1-\sigma} + \lambda_2 {w_2}^{1-\sigma}]^{\frac{1}{1-\sigma}}$$

$$w_1 = [Y_1 {G_1}^{\sigma-1} + Y_2 {G_2}^{\sigma-1} T^{1-\sigma}]^{\frac{1}{\sigma}}$$

$$w_2 = [Y_1 {G_1}^{\sigma-1} T^{1-\sigma} + Y_2 {G_2}^{\sigma-1}]^{\frac{1}{\sigma}}$$

$$\omega_1 = w_1 {G_1}^{-\mu}, \qquad \omega_2 = w_2 {G_2}^{-\mu}$$

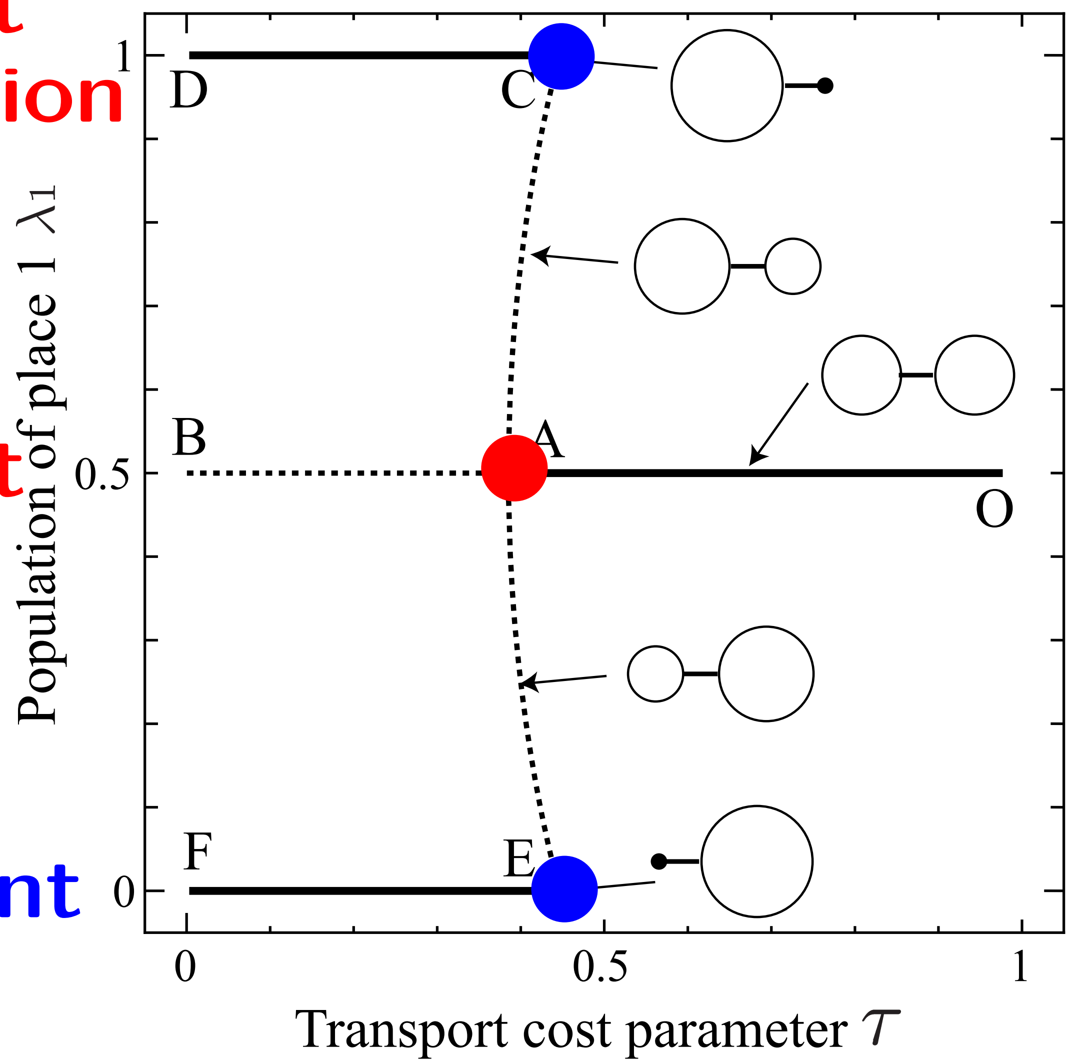
$$\frac{d\lambda_1}{dt} = (\omega_1(\lambda,\tau) - \omega_2(\lambda,\tau))\lambda_1\lambda_2$$

Two-Place Economy

Low transport cost causes agglomeration

τ_A: **break point**

τ_E: **sustain point**



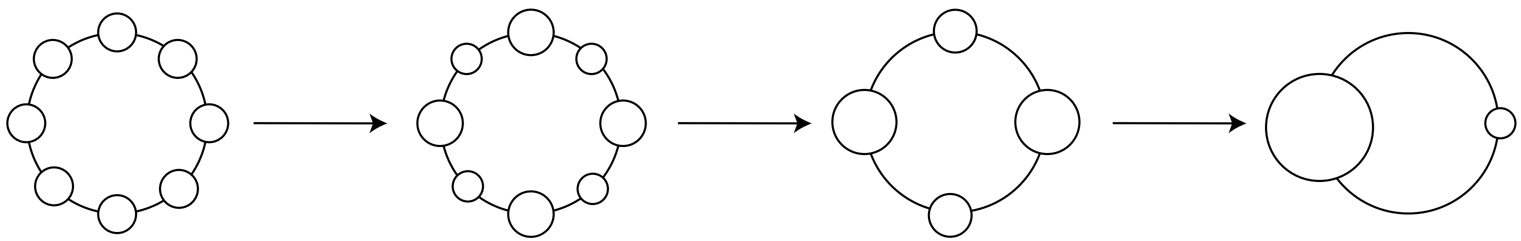
New Econ. Geography: State-of-the-Art

- Micro-economic model:

  core-periphery → refinements

- Spatial platform:

  two-place → long narrow, racetrack



Krugman (1996): The Self-organizing Economy

I have demonstrated the emergence of a regular lattice only for a one-dimensional economy, but I have no doubt that a better mathematician could show that a system of hexagonal market areas will emerge in two dimensions.

Long Narrow Economy

Fujita, Mori (1997): Regional Sci Urban Econon

Structural stability and evolution of urban systems

Fujita, Krugman, Mori (1999): Euro Econ Review

On the evolution of hierarchical urban systems

Racetrack Economy

Krugman (1993): Euro Econ Review

On the number and location of cities

Mossay (2003): Regional Sci Urban Econ

Increasing returns and heterogeneity in a spatial economy

Picard, Tabuchi (2010): Economic Theory

Self-organized agglomerations and transport costs

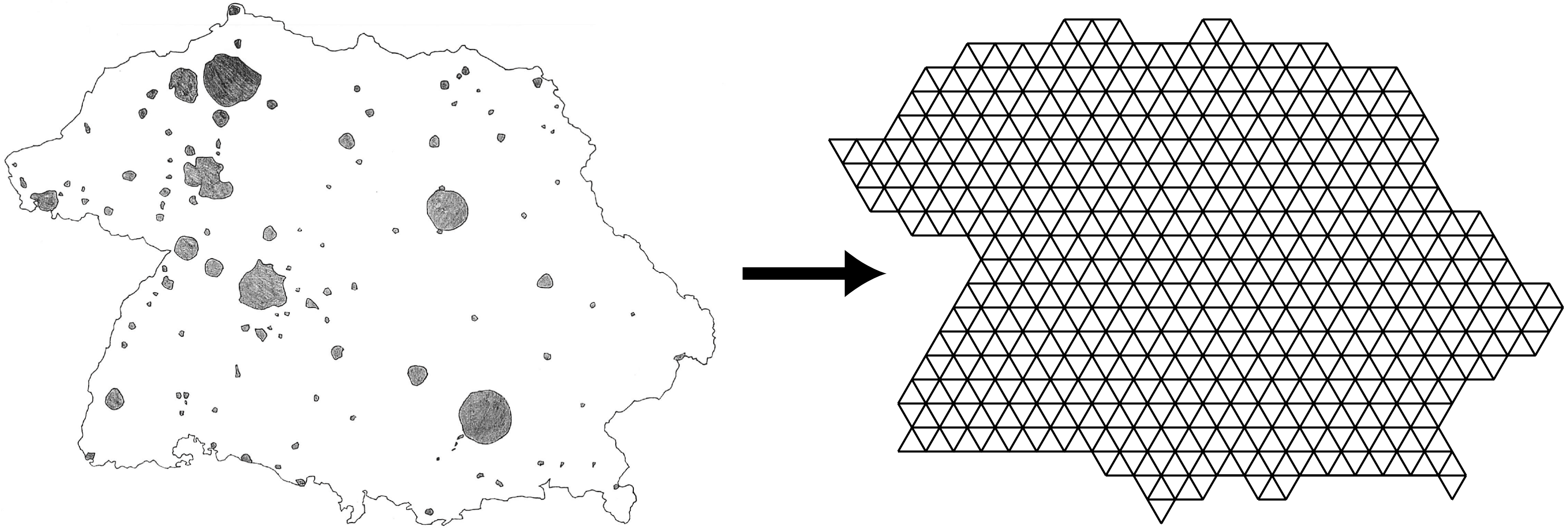
Tabuchi, Thisse (2011): J. Urban Economics

A new economic geography model of central places

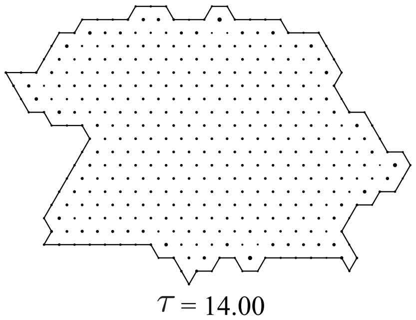
Part 2.

Hexagonal Agglomeration

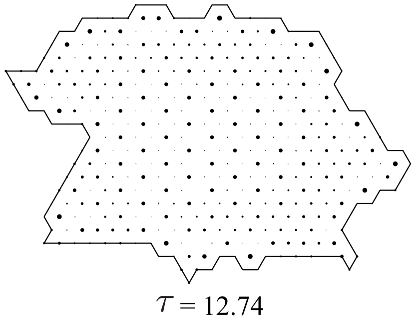
Discretization for Southern Germany



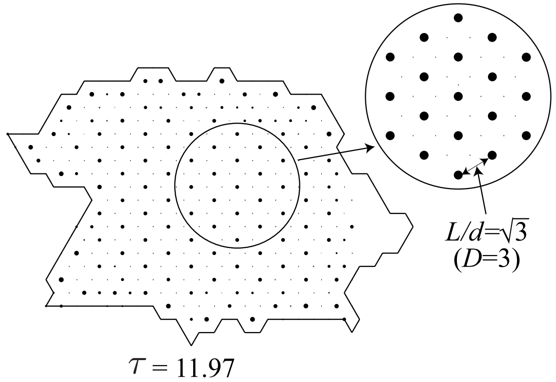
Initial Stages (high transport cost)



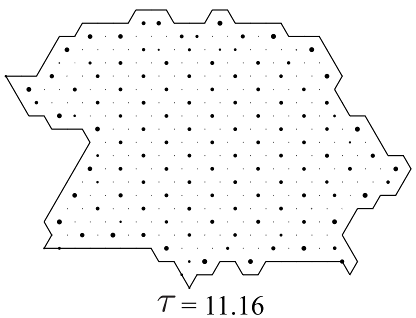
A



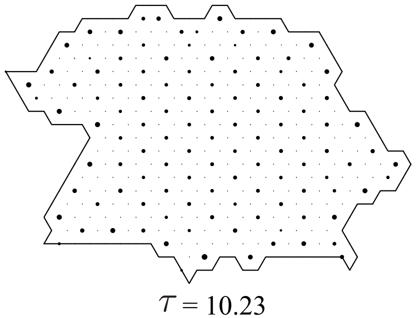
B



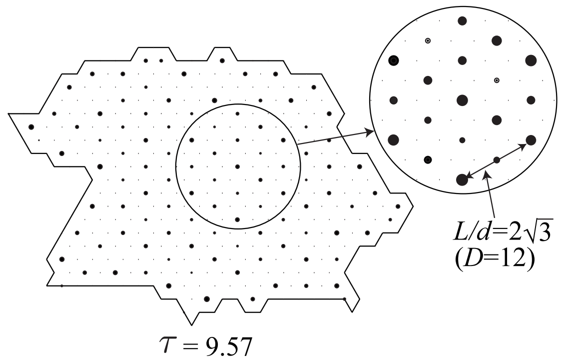
C $(l/d = \sqrt{3})$



D

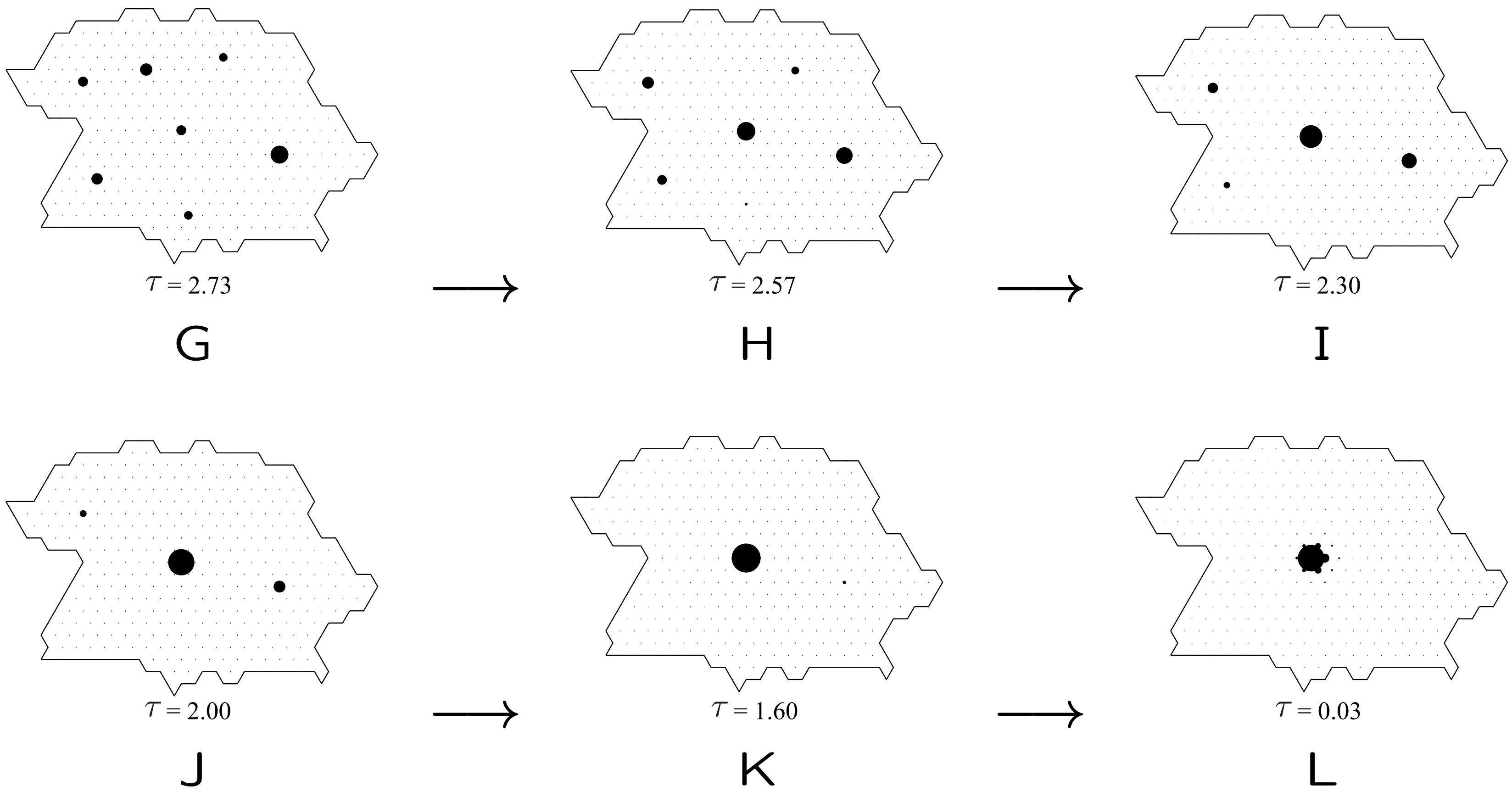


E



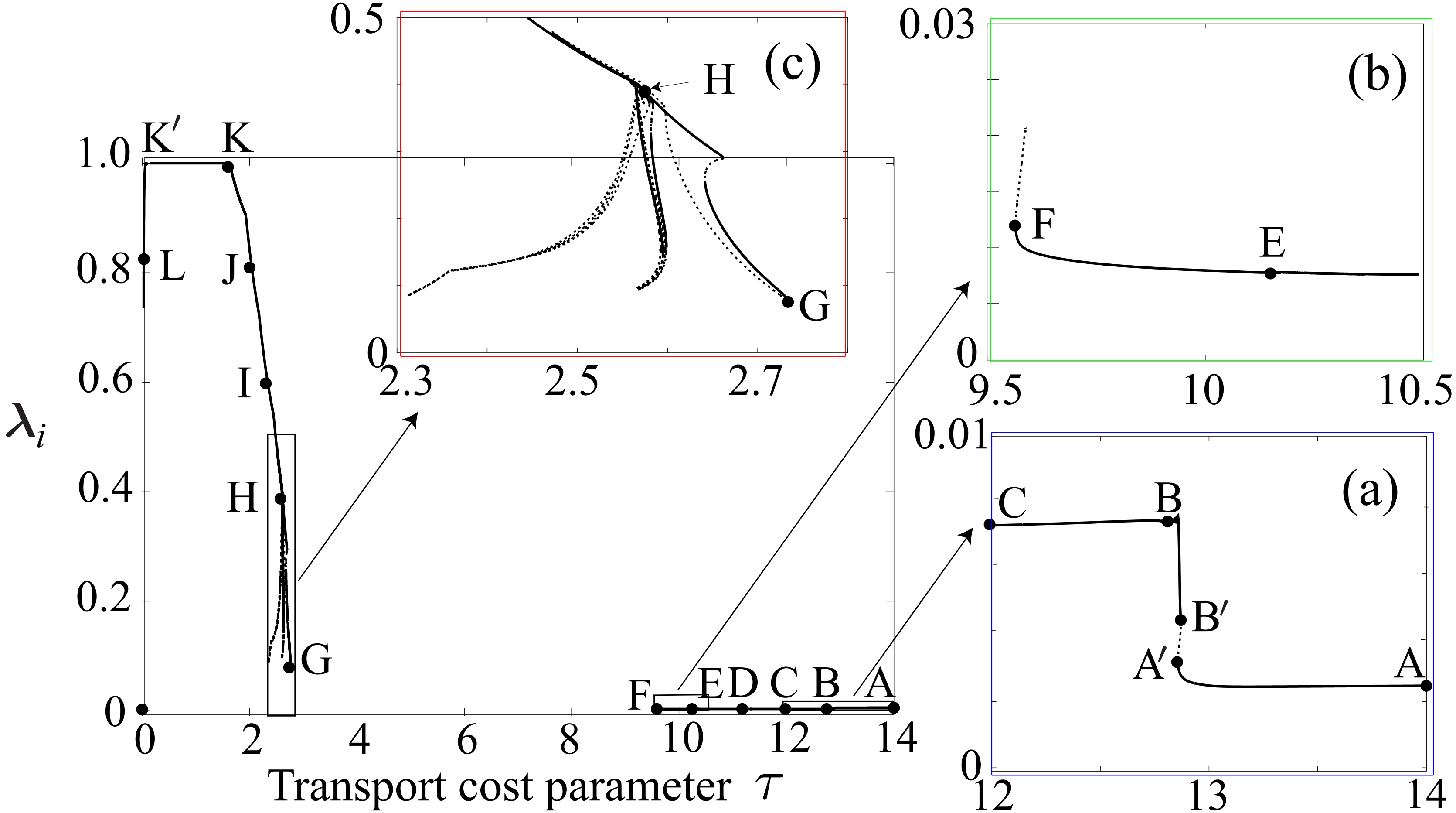
F $(l/d = 2\sqrt{3})$

Final Stages (low transport cost)



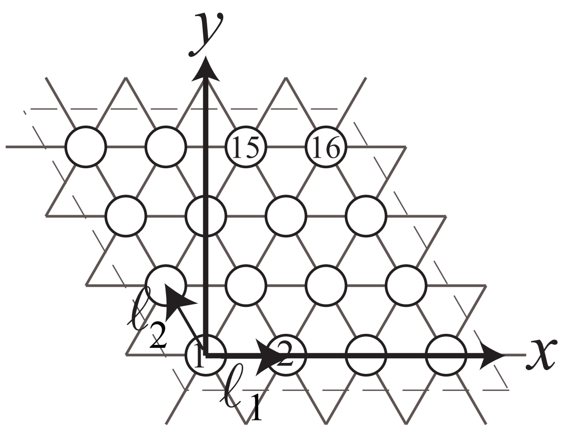
Numerical Analysis for Southern Germany

population vs transport cost

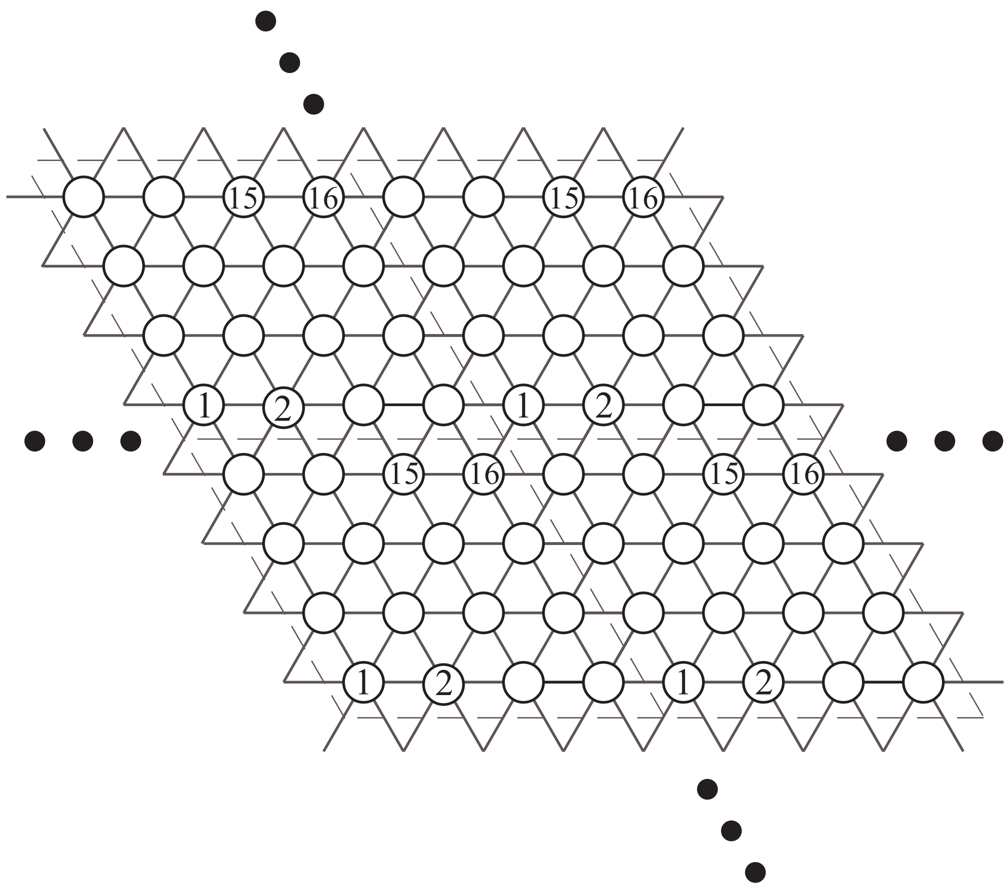


(Forslid–Ottaviano model (2003), logit choice function)

Modeling by Periodic Finite Hexagonal Lattice



(a) 4×4 lattice



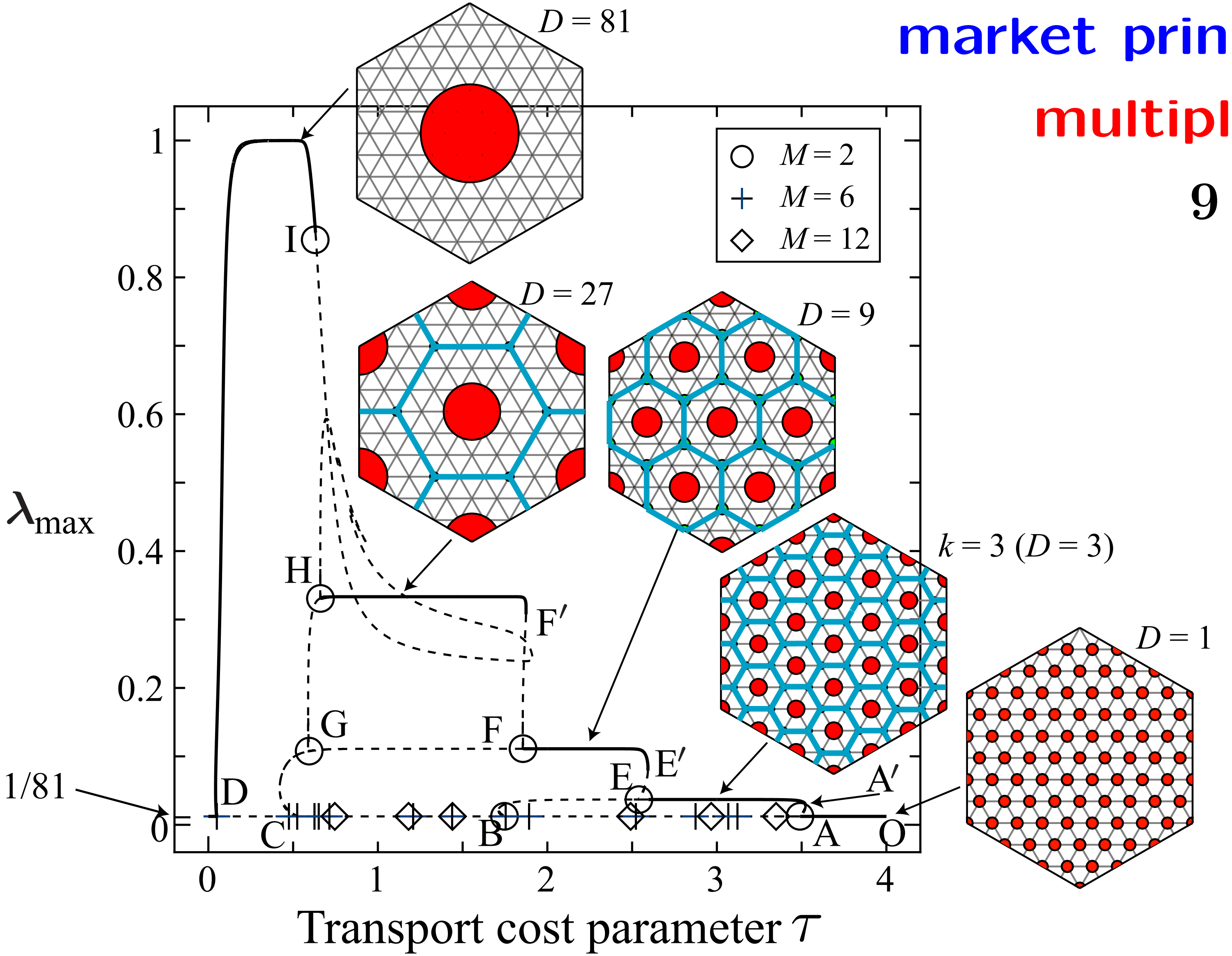
(b) Periodically repeated

Emergence of Central Places (1)

market principle $k = 3$

multiplicity $M = 2$

9×9 lattice



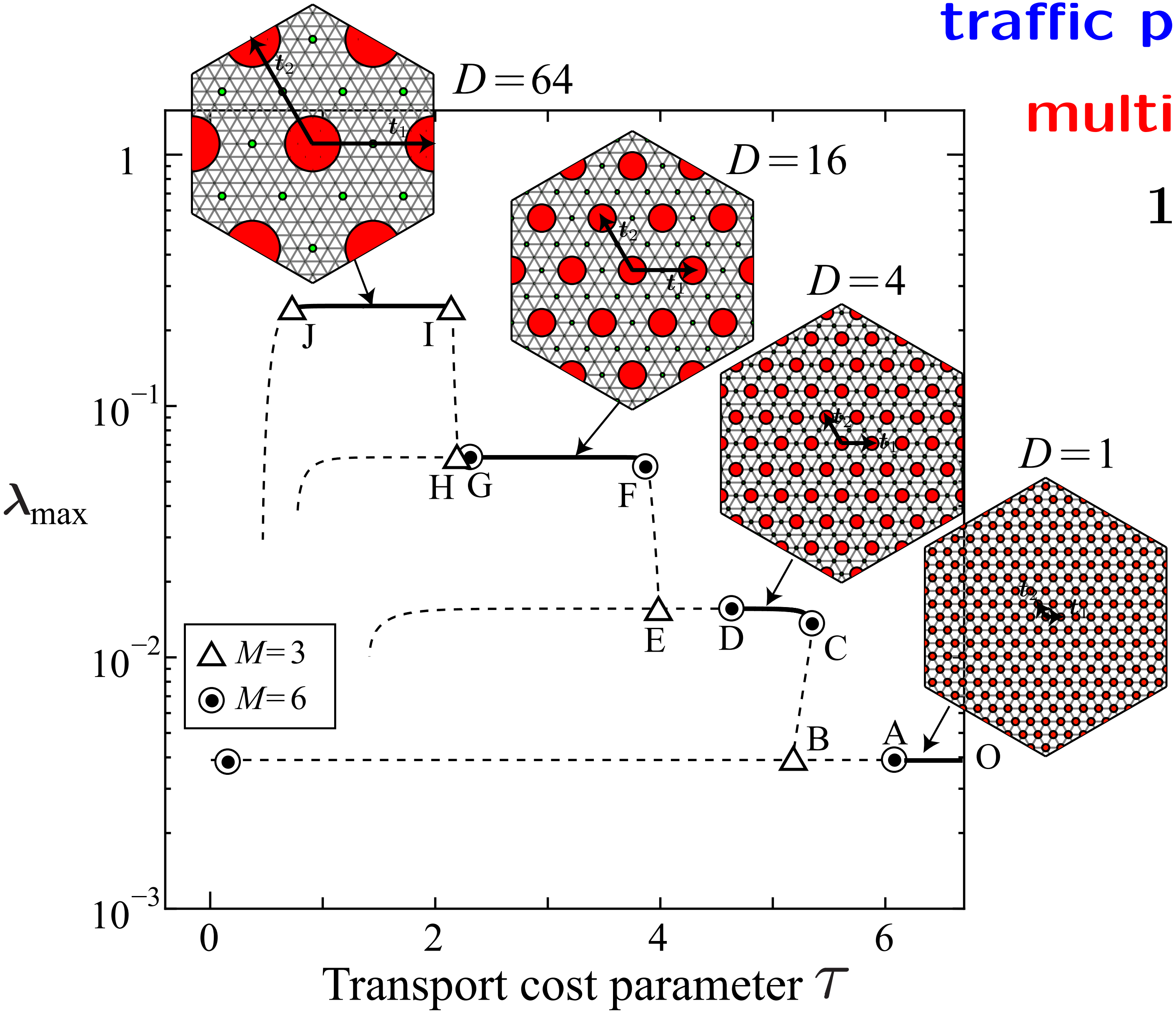
(Forslid–Ottaviano model (2003), logit choice function)

Emergence of Central Places (2)

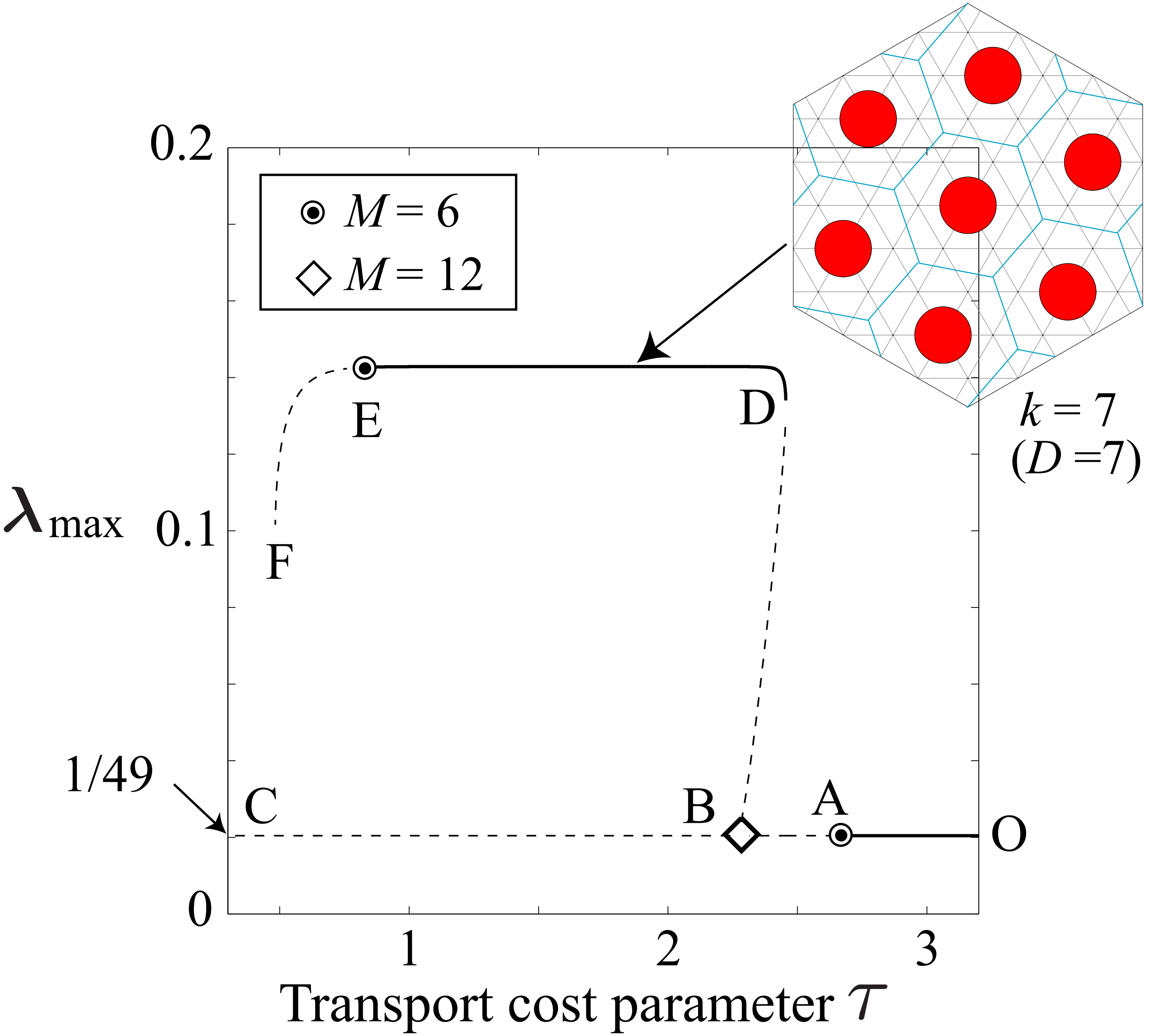
traffic principle $k = 4$

multiplicity $M = 3$

16×16 lattice



Emergence of Central Places (3)



administrative principle $k = 7$

multiplicity $M = 12$

7×7 lattice

Summary of Our Results

Christaller's	size n	Mult M
$k = 3$ (market)	$3\times$	2
$k = 4$ (traffic)	$2\times$	3
$k = 7$ (administrative)	$7\times$	12

Lösch's D	size n	Mult M
9 (traffic-like)	$3\times$	6
12 (market-like)	$6\times$	6
13 (admin-like)	$13\times$	12
16 (traffic-like)	$4\times$	6
19 (admin-like)	$19\times$	12
21 (admin-like)	$21\times$	12
25 (traffic-like)	$5\times$	6

Lattice Economy

- **Ikeda, Murota, Akamatsu, Kono, Takayama, Sobhaninejad, Shibasaki (2010):**

Self-organizing hexagons in economic agglomeration: core-periphery models and central place theory, METR 2010-28, U. Tokyo.

Discovery of hexagonal patterns (numerical, theoretical)

- **Takayama, Akamatsu (2010):** 土木計画学研究・論文集.
- **Ikeda, Murota, Akamatsu (2012):** Self-organization of Lösch's hexagons in economic $\cdots$, Int. J. Bifurcation & Chaos.
- **Ikeda, Murota, Akamatsu, Kono, Takayama (2014):** Self-organization of hexagonal $\cdots$, J. Economic Behav. & Organiz.

- **Ikeda, Murota (2014):**

Bifurcation Theory for Hexagonal Agglomeration in Economic Geography.

Systematic presentation of the theory

Part 3.

Group-Theoretic Bifurcation Theory

Group-theoretic Bifurcation Theory

- **Sattinger (1979):**

Group Theoretic Methods in Bifurcation Theory.
(Lecture Notes in Mathematics)

- **Golubitsky, Schaeffer (1985):**

Singularities and Groups in Bifurcation Theory, Vol. 1

- **Golubitsky, Stewart, Schaeffer (1988):**

Singularities and Groups in Bifurcation Theory, Vol. 2

Bifurcation Analysis of Two-Place Economy

poplutation $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, **transport cost $\boldsymbol{\tau}$**

$$\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \begin{bmatrix} F_1(\lambda_1, \lambda_2, \tau) \\ F_2(\lambda_1, \lambda_2, \tau) \end{bmatrix} = \boldsymbol{0}$$

$$\begin{aligned} F_1(\lambda_1, \lambda_2, \tau) &= (\omega_1(\lambda_1, \lambda_2, \tau) - \overline{\omega}(\lambda_1, \lambda_2, \tau))\lambda_1 \\ F_2(\lambda_1, \lambda_2, \tau) &= (\omega_2(\lambda_1, \lambda_2, \tau) - \overline{\omega}(\lambda_1, \lambda_2, \tau))\lambda_2 \end{aligned}$$

average real wage $\overline{\omega}(\lambda_1, \lambda_2, \tau) = \lambda_1\omega_1 + \lambda_2\omega_2$

Symmetry: $F_2(\lambda_1, \lambda_2, \tau) = F_1(\lambda_2, \lambda_1, \tau)$

Formulation of Symmetry

Symmetry: $F_2(\lambda_1, \lambda_2, \tau) = F_1(\lambda_2, \lambda_1, \tau)$

$\Longleftrightarrow$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1(\lambda, \tau) \\ F_2(\lambda, \tau) \end{bmatrix} = F\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \lambda, \tau \right)$$

$\Longleftrightarrow$

Equivariance: $T(g)F(\lambda, \tau) = F(T(g)\lambda, \tau), \quad g \in G$

$G = \{e, s\}$: group, $\quad s : (1, 2) \mapsto (2, 1)$,

$$T(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reduction to Bifurcation Equation

Critical point $(\lambda_c, \tau_c) = (1/2, 1/2, \tau_c)$ at some $\tau = \tau_c$

New variable $w = \lambda_1 - \lambda_2$; $\widetilde{f} = \tau - \tau_c$

$$\lambda_1 = \frac{1+w}{2}, \qquad \lambda_2 = \frac{1-w}{2}$$

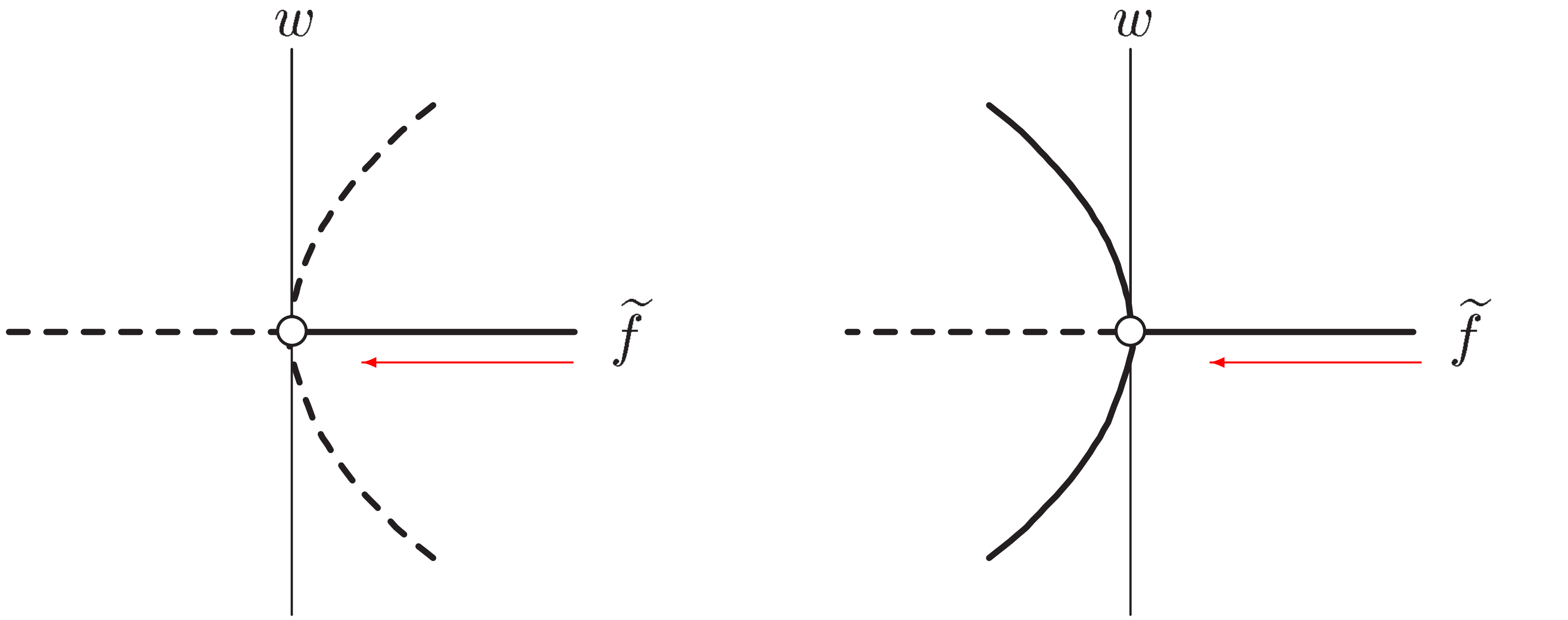
$\Downarrow$ **Bifurcation equation:**

$$\begin{aligned} \widetilde{F}(w, \widetilde{f}) &= F_1\left(\frac{1+w}{2}, \frac{1-w}{2}, \widetilde{f}\right) - F_2\left(\frac{1+w}{2}, \frac{1-w}{2}, \widetilde{f}\right) \\ &= w[A\,\widetilde{f} + Bw^2 + \cdots] = 0 \end{aligned}$$

$\Downarrow$ **Two kinds of solutions (equilibria):**

$$\begin{cases} w = 0, & \text{trivial equilibria } (\lambda_1 = \lambda_2), \\ \widetilde{f} = -\frac{B}{A}w^2 + \cdots & \text{bifurcating equilibria } (\lambda_1 \neq \lambda_2) \end{cases}$$

Pitchfork Bifurcation



subcritical $(A < 0,\ B > 0)$

supercritical $(A < 0,\ B < 0)$

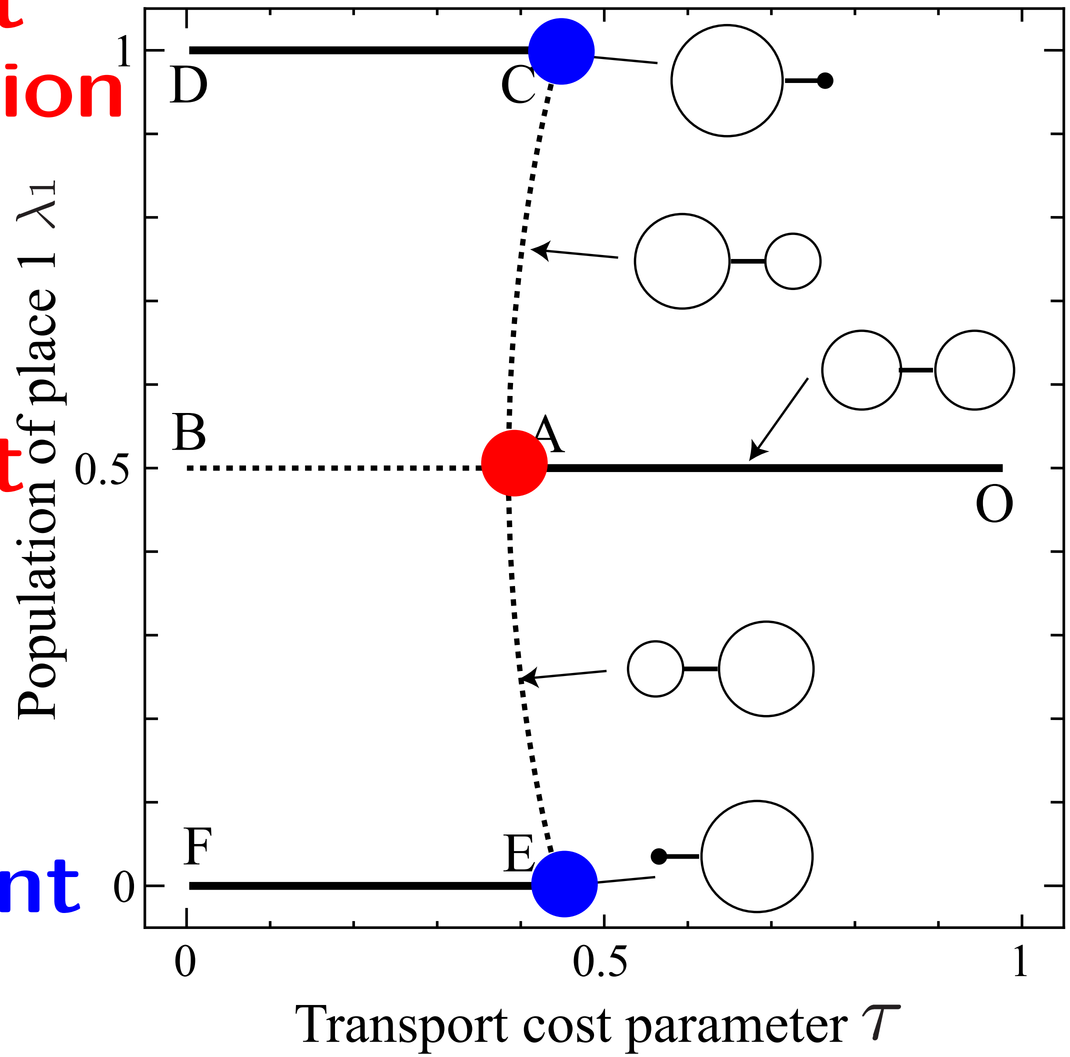
○: **bifurcation point,** —: **stable,** – – –: **unstable**

Two-Place Economy

Low transport cost causes agglomeration

τ_A: break point

τ_E: sustain point



Methodological Characteristics

Group-theoretic method shows:

(1) Reduction to bifurcation equation
dimension (# vars/eqns), choice of vars

(2) Possible bifurcating equilibria
symmetry/pattern (e.g., Christaller's systems)

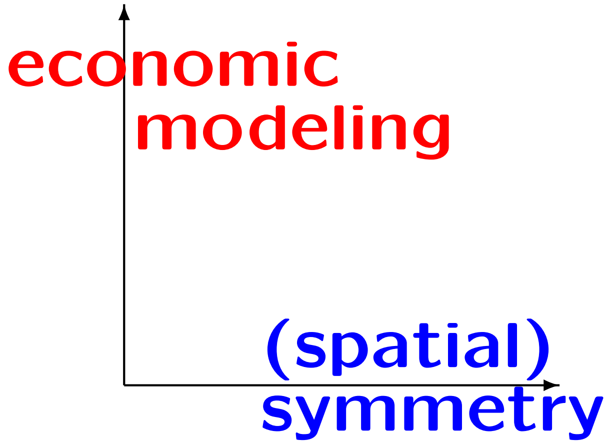
(3) Generic (structural) properties under symmetry, independent of individual models and parameters
structural degeneracy vs accidental coincidence

Does not capture:

(1) Specific value of τ_c

(2) Specific values of A, B, etc.

(3) Stability of equilibria



Equivariance for Hexagonal Lattice

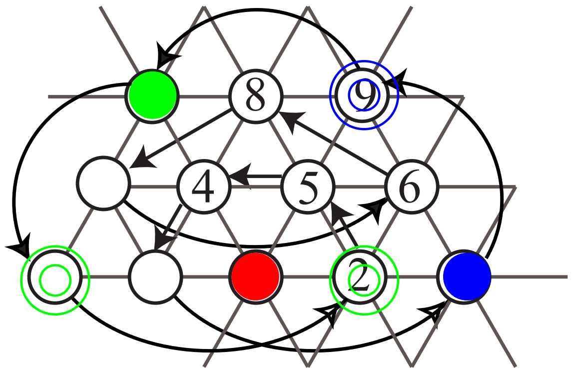
$$\boldsymbol{T(g)F(\lambda,\tau) = F(T(g)\lambda,\tau)}, \qquad \boldsymbol{g \in G}$$

$$\boldsymbol{G =} \quad \cdots$$

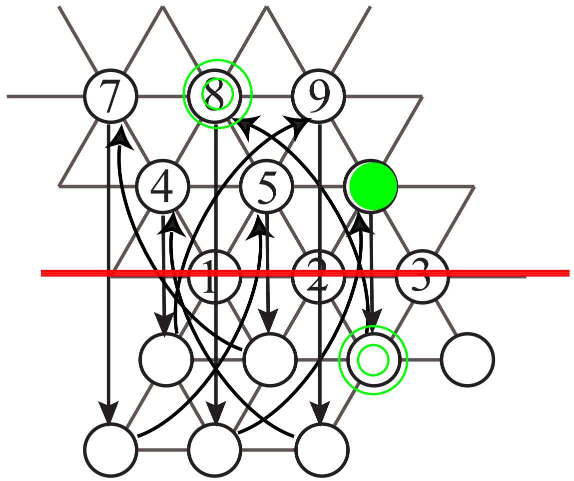
$$\boldsymbol{T(g) =} \quad \cdots$$

Symmetry of 3 x 3 Lattice

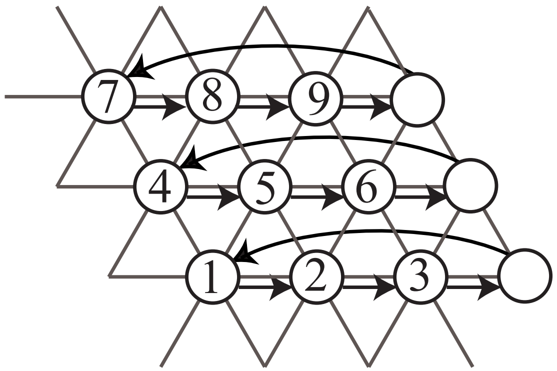
$G = \langle r, s, p_1, p_2 \rangle$



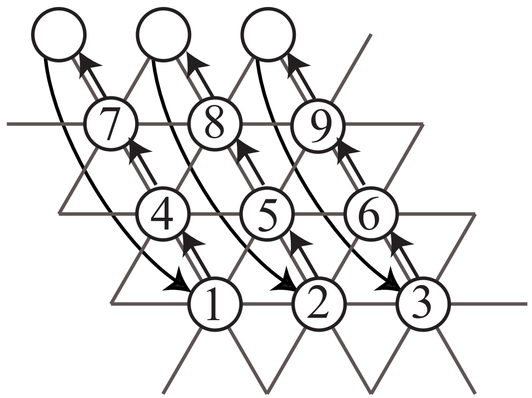
rotation r



reflection s



translation p_1



translation p_2

Representation Matrices T $(n = 3)$

$$
r \mapsto \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & & & & 1 & & \\
 & & & 1 & & & & & \\
\hline
 & & & & 1 & & & & \\
 & 1 & & & & & & & \\
 & & & & & & & 1 & \\
\hline
 & & & & & & & & 1 \\
 & & & & & 1 & & & \\
 & & 1 & & & & & &
\end{array}\right], \qquad
s \mapsto \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
\hline
 & & & & & & & & 1 \\
 & & & & & & 1 & & \\
 & & & & & & & 1 & \\
\hline
 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & 1 & & & & &
\end{array}\right]
$$

$$
p_1 \mapsto \left[\begin{array}{ccc|ccc|ccc}
 & & 1 & & & & & & \\
1 & & & & & & & & \\
 & 1 & & & & & & & \\
\hline
 & & & & & 1 & & & \\
 & & & 1 & & & & & \\
 & & & & 1 & & & & \\
\hline
 & & & & & & & & 1 \\
 & & & & & & 1 & & \\
 & & & & & & & 1 &
\end{array}\right], \qquad
p_2 \mapsto \left[\begin{array}{ccc|ccc|ccc}
 & & & & & & 1 & & \\
 & & & & & & & 1 & \\
 & & & & & & & & 1 \\
\hline
1 & & & & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
\hline
 & & & 1 & & & & & \\
 & & & & 1 & & & & \\
 & & & & & 1 & & &
\end{array}\right]
$$

$$G = \langle r, s, p_1, p_2 \rangle$$

Equivariance for Hexagonal Lattice

$$\boldsymbol{T(g)F(\lambda,\tau) = F(T(g)\lambda,\tau), \qquad g \in G}$$

$$\boldsymbol{G = \langle r, s, p_1, p_2 \rangle}$$

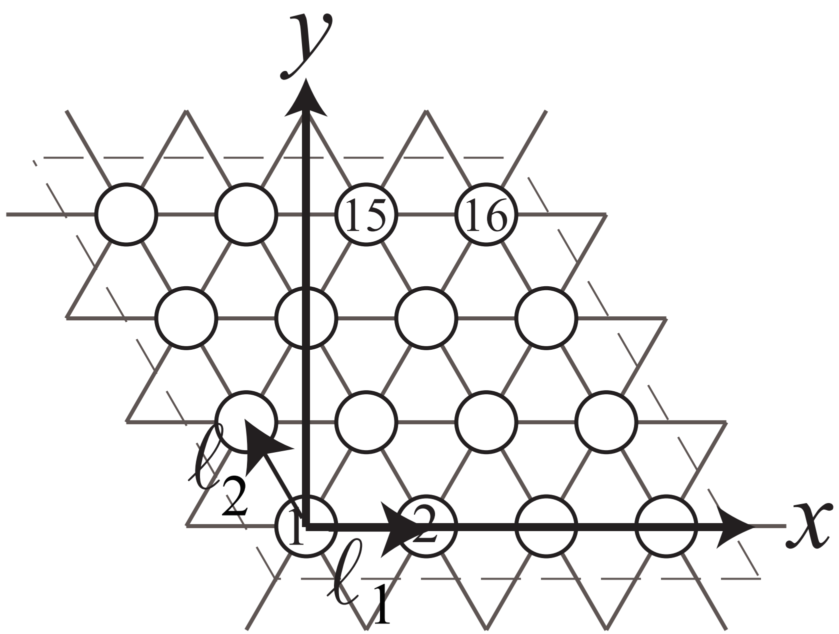
$$\boldsymbol{T(g)} = \left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & & & & 1 & & \\
 & & & 1 & & & & & \\
\hline
 & & & & 1 & & & & \\
 & 1 & & & & & & & \\
 & & & & & & & 1 & \\
\hline
 & & & & & & & & 1 \\
 & & & & & 1 & & & \\
 & & 1 & & & & & &
\end{array}\right] \quad \textbf{(etc.)}$$

Symmetry of $n \times n$ Lattice

r: rotation ($\pi/3$ rad)

s: reflection

p_1, p_2: translations



$$G = \langle r, s, p_1, p_2 \rangle$$
$$= \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$$

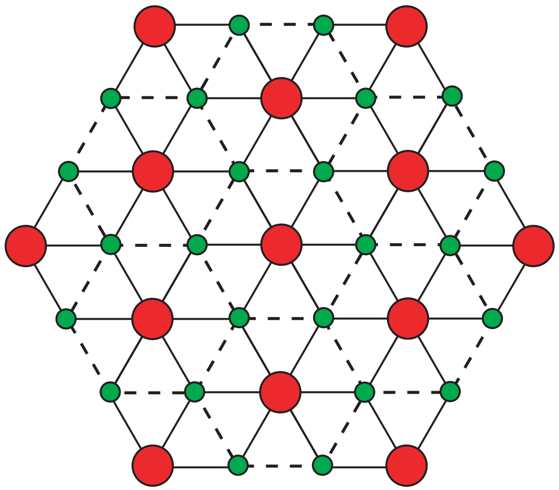
$$r^6 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e, \quad p_2 p_1 = p_1 p_2,$$
$$r p_1 = p_1 p_2 r, \quad r p_2 = p_1^{-1} r, \quad s p_1 = p_1 s, \quad s p_2 = p_1^{-1} p_2^{-1} s$$

Subgroups for Christaller's Systems

Symmetry: $G = \langle r, s, p_1, p_2 \rangle = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

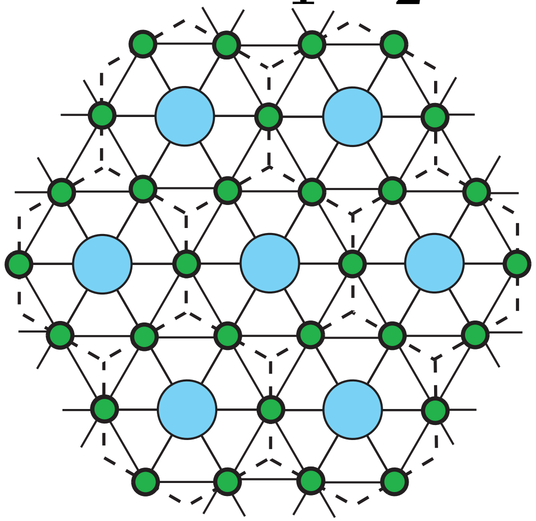
Partial Symmetry:

$\langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$



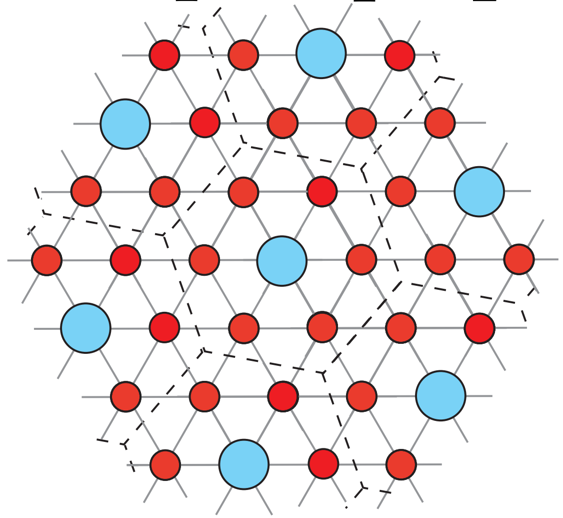
$k = 3$

$\langle r, s, p_1^2, p_2^2 \rangle$



$k = 4$

$\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$



$k = 7$

Reduction to Bifurcation Equation

Liapunov-Schmidt reduction

eliminates variables by implicit function thm

Which variables remain?

$J_{\rm c}$: Jacobian at critical point

dim Ker($J_{\rm c}$) = dim of bifur.eqn (# eqns/vars)

Ker($J_{\rm c}$): invariant subspace $\longleftrightarrow$ irred representation (generically)

dim bifur.eqn = dim irred rep in T

= 2, 3, 6, 12 [NOT: 4]

Bifurcation eqn: $\tilde{F}(\lambda, \tau) = 0$

Equivariance: $\tilde{T}(g)\tilde{F}(\lambda, \tau) = \tilde{F}(\tilde{T}(g)\lambda, \tau), \quad g \in G$

Group Representation

Representation of G is a mapping $T : G \to \mathrm{GL}(N, \mathbb{R})$:

$$T(gh) = T(g)T(h), \quad g, h \in G.$$

Invariant subspace: $w \in W \Rightarrow T(g)w \in W \ (\forall g \in G)$

Irreducible rep: does not have invariant subspaces

A finite family determined by G

Decomposition into irred reps: (essent.) unique for T

$$Q^{-1}TQ = T^{(1)} \oplus T^{(2)} \oplus T^{(3)} \oplus \cdots$$

Irreducible Decomposition ($n = 3$)

$$
Q^{-1}T(g)Q = Q^{-1}\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & & & & & & 1 & & \\
 & & & 1 & & & & & \\
\hline
 & & & & 1 & & & & \\
 & 1 & & & & & & & \\
 & & & & & & & 1 & \\
\hline
 & & & & & & & & 1 \\
 & & & & & 1 & & & \\
 & & 1 & & & & & &
\end{array}\right] Q
$$

$$
= \left[\begin{array}{c|cc|cccccc}
* & & & & & & & & \\
\hline
 & * & * & & & & & & \\
 & * & * & & & & & & \\
\hline
 & & & * & * & * & * & * & * \\
 & & & * & * & * & * & * & * \\
 & & & * & * & * & * & * & * \\
 & & & * & * & * & * & * & * \\
 & & & * & * & * & * & * & * \\
 & & & * & * & * & * & * & *
\end{array}\right]
$$

dim: $1 + 2 + 6$

Procedure of Group-th. Bifurcation Analysis

- Find **symmetry group** G & **representation** T
- Enumerate all **irred reps** μ of G

1,2,3,4,6,12-dim (by method of little groups)

- **Decompose** T into irred reps μ
- For each irred rep μ:

  A: Derive and solve **bifurcation eqn**
  to find bifur. solution and see the symmetry

  B: Apply **equivariant branching lemma**
  to see the existence of specified symmetry

Irreducible Representations of $G = \langle r, s, p_1, p_2 \rangle$

n	dim**1**	dim**2**	dim**3**	dim**4**	dim **6**	dim **12**
$6m$	**4**	**4**	**4**	**1**	$2n-6$	$(n^2-6n+12)/12$
$6m \pm 1$	**4**	**2**	**0**	**0**	$2n-2$	$(n^2-6n+5)/12$
$6m \pm 2$	**4**	**2**	**4**	**0**	$2n-4$	$(n^2-6n+8)/12$
$6m \pm 3$	**4**	**4**	**0**	**1**	$2n-4$	$(n^2-6n+9)/12$

- **dim 6 exist for** $n \geq 3$
- **dim 12 exist for** $n \geq 6$

n	dim**1**	dim**2**	dim**3**	dim**4**	dim **6**	dim **12**
3	**4**	**4**	**0**	**1**	2	0
6	**4**	**4**	**4**	**1**	6	1
7	**4**	**2**	**0**	**0**	2	1

3-dim Irreducible Rep

$$\begin{aligned} r &: (w_1, w_2, w_3) \mapsto (w_3, w_1, w_2) \\ s &: (w_1, w_2, w_3) \mapsto (w_3, w_2, w_1) \\ p_1 &: (w_1, w_2, w_3) \mapsto (-w_1, w_2, -w_3) \\ p_2 &: (w_1, w_2, w_3) \mapsto (w_1, -w_2, -w_3) \end{aligned}$$

$$T(r) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \qquad T(s) = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

$$T(p_1) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \qquad T(p_2) = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

$$(3; +, +)$$

Bifurcation Equations for $M = 3$ (1)

$$F_i(w_1, w_2, w_3, \widetilde{\tau}) = 0 \quad (i = 1, 2, 3)$$

Equivariance conditions:

$$\begin{aligned}
r: \quad & F_3(w_1, w_2, w_3) = F_1(w_3, w_1, w_2) \\
& F_1(w_1, w_2, w_3) = F_2(w_3, w_1, w_2) \\
& F_2(w_1, w_2, w_3) = F_3(w_3, w_1, w_2) \\
s: \quad & F_3(w_1, w_2, w_3) = F_1(w_3, w_2, w_1) \\
& F_2(w_1, w_2, w_3) = F_2(w_3, w_2, w_1) \\
& F_1(w_1, w_2, w_3) = F_3(w_3, w_2, w_1) \\
p_1: \quad & -F_1(w_1, w_2, w_3) = F_1(-w_1, w_2, -w_3) \\
& F_2(w_1, w_2, w_3) = F_2(-w_1, w_2, -w_3) \\
& -F_3(w_1, w_2, w_3) = F_3(-w_1, w_2, -w_3) \\
p_2: \quad & F_1(w_1, w_2, w_3) = F_1(w_1, -w_2, -w_3) \\
& -F_2(w_1, w_2, w_3) = F_2(w_1, -w_2, -w_3) \\
& -F_3(w_1, w_2, w_3) = F_3(w_1, -w_2, -w_3)
\end{aligned}$$

Bifurcation Equations for $M = 3$ (2)

Conditions connecting F_2 to (F_1, F_3):

$$F_1(w_1, w_2, w_3) = F_2(w_3, w_1, w_2)$$

$$F_3(w_1, w_2, w_3) = F_2(w_2, w_3, w_1)$$

Conditions on F_2:

$$F_2(w_1, w_2, w_3) = F_2(-w_1, w_2, -w_3)$$

$$-F_2(w_1, w_2, w_3) = F_2(w_1, -w_2, -w_3)$$

$$F_2(w_1, w_2, w_3) = F_2(w_3, w_2, w_1)$$

$$\Downarrow$$

$$F_2 = w_2 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a,2b+1,2c}(\widetilde{\tau})\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c} + w_1 w_3 \sum_{a=0} \sum_{b=0} \sum_{c=0} A_{2a+1,2b,2c+1}(\widetilde{\tau})\, {w_1}^{2a} {w_2}^{2b} {w_3}^{2c}$$

Bifurcation Equations for $M = 3$ (3)

$$
\begin{aligned}
F_2 &= w_2 \sum\sum\sum A_{2a,2b+1,2c}(\widetilde{\tau})\, {w_1}^{2a}{w_2}^{2b}{w_3}^{2c} \\
&\quad + w_1 w_3 \sum\sum\sum A_{2a+1,2b,2c+1}(\widetilde{\tau})\, {w_1}^{2a}{w_2}^{2b}{w_3}^{2c} \\
F_1 &= F_2(w_3, w_1, w_2), \qquad F_3 = F_2(w_2, w_3, w_1)
\end{aligned}
$$

Trivial solution: $w_1 = w_2 = w_3 = 0$

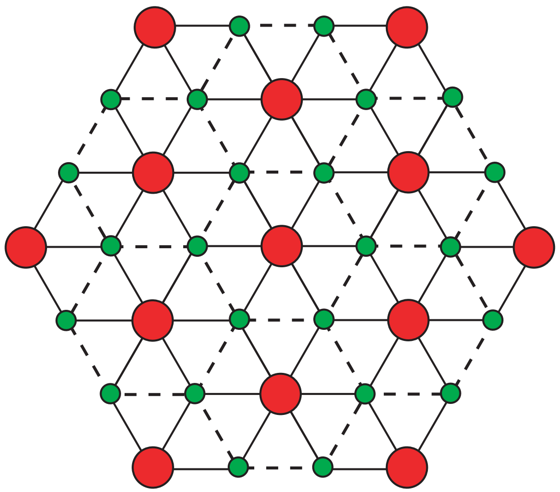
Bifurcating solution: $w_1 = w_2 = w_3 \neq 0$

$$
\begin{aligned}
0 &= \sum\sum\sum A_{2a,2b+1,2c}(\widetilde{\tau})\, {w_1}^{2(a+b+c)} \\
&\quad + w_1 \sum\sum\sum A_{2a+1,2b,2c+1}(\widetilde{\tau})\, {w_1}^{2(a+b+c)} \\
&\approx A\widetilde{\tau} + Bw_1 \\
&\longrightarrow w_1 \approx -(A/B)\widetilde{\tau}
\end{aligned}
$$

Bifurcation Equations for $M = 3$ (4)

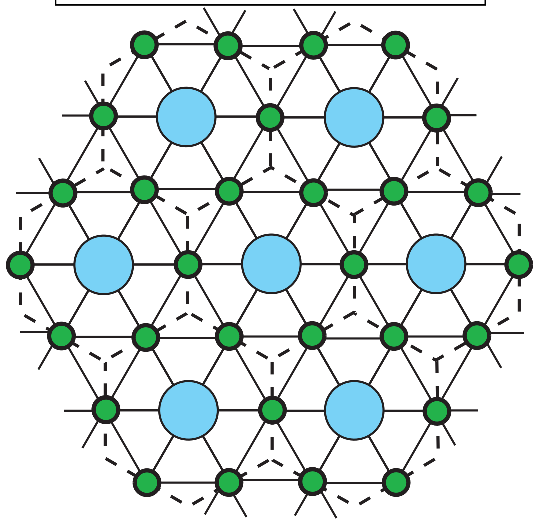
Symmetry of $(w, w, w) = \langle r, s, p_1^2, p_2^2 \rangle$

$\langle r, s, p_1^2 p_2, p_1^{-1} p_2 \rangle$



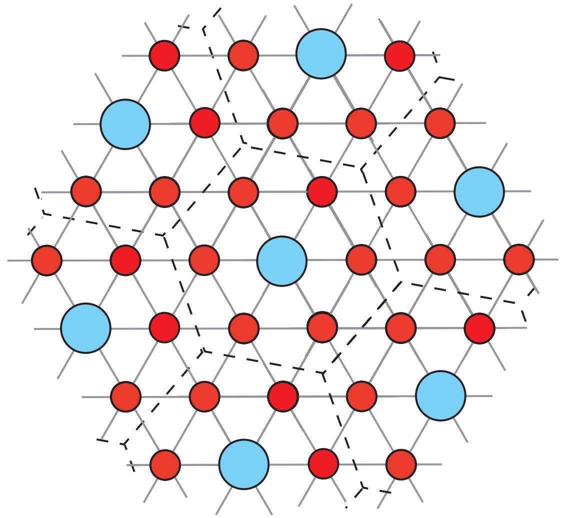
$k = 3$

$\boxed{\langle r, s, p_1^2, p_2^2 \rangle}$



$k = 4$

$\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$



$k = 7$

Bifurcation Equations for $M = 3$ (5)

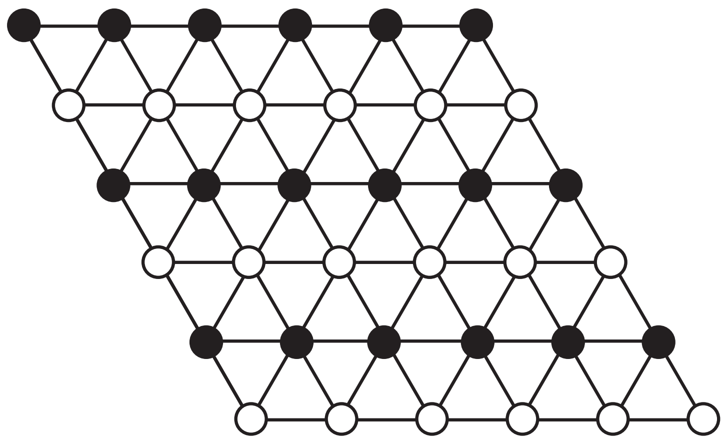
$$F_2 = w_2 \sum\sum\sum A_{2a,2b+1,2c}(\tilde{\tau})\, {w_1}^{2a}{w_2}^{2b}{w_3}^{2c} + w_1 w_3 \sum\sum\sum A_{2a+1,2b,2c+1}(\tilde{\tau})\, {w_1}^{2a}{w_2}^{2b}{w_3}^{2c}$$

$$F_1 = F_2(w_3, w_1, w_2), \qquad F_3 = F_2(w_2, w_3, w_1)$$

Another bifurcating solution: $w_2 \neq 0,\ w_1 = w_3 = 0$

$$0 = \sum A_{0,2b+1,0}(\tilde{\tau})\, {w_2}^{2b} \approx A\tilde{\tau} + B{w_2}^2$$

$$\longrightarrow \tilde{\tau} \approx -(B/A)B{w_2}^2$$



$\langle r^3, s, p_1, p_2^2 \rangle$

12-dim Irreducible Rep

$(12; k, \ell) \qquad (1 \le \ell \le k-1,\ 2k+\ell \le n-1)$

$$r \mapsto \left[\begin{array}{ccc|ccc} & & S & & & \\ S & & & & & \\ & S & & & & \\ \hline & & & & S & \\ & & & & & S \\ & & & S & & \end{array}\right], \qquad s \mapsto \left[\begin{array}{ccc|ccc} & & & I & & \\ & & & & I & \\ & & & & & I \\ \hline I & & & & & \\ & I & & & & \\ & & I & & & \end{array}\right]$$

$p_1 \mapsto$ $\qquad$ $p_2 \mapsto$

$$\left[\begin{array}{ccc|ccc} R^k & & & & & \\ & R^\ell & & & & \\ & & R^{-k-\ell} & & & \\ \hline & & & R^k & & \\ & & & & R^\ell & \\ & & & & & R^{-k-\ell} \end{array}\right], \left[\begin{array}{ccc|ccc} R^\ell & & & & & \\ & R^{-k-\ell} & & & & \\ & & R^k & & & \\ \hline & & & R^{-k-\ell} & & \\ & & & & R^k & \\ & & & & & R^\ell \end{array}\right]$$

$$R = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

12-dim Irreducible Rep (complex variables)

$(12; k, \ell) \qquad (1 \le \ell \le k-1,\ 2k+\ell \le n-1)$

$$r: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z}_3 \\ \overline{z}_1 \\ \overline{z}_2 \\ \overline{z}_5 \\ \overline{z}_6 \\ \overline{z}_4 \end{bmatrix} \qquad s: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} z_4 \\ z_5 \\ z_6 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$p_1: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^k\ z_1 \\ \omega^\ell\ z_2 \\ \omega^{-k-\ell}\ z_3 \\ \omega^k\ z_4 \\ \omega^\ell\ z_5 \\ \omega^{-k-\ell}\ z_6 \end{bmatrix} \qquad p_2: \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} \mapsto \begin{bmatrix} \omega^\ell\ z_1 \\ \omega^{-k-\ell}\ z_2 \\ \omega^k\ z_3 \\ \omega^{-k-\ell}\ z_4 \\ \omega^k\ z_5 \\ \omega^\ell\ z_6 \end{bmatrix}$$

$\omega = \exp(\mathrm{i}2\pi/n)$

Bifurcation Equations for $M = 12$ (1)

$$F_i(z_1,\ldots,z_6) = 0, \quad i = 1,\ldots,6; \qquad z_j \in \mathrm{C}$$

$$\begin{aligned}
r: \quad & \overline{F_3(z_1,z_2,z_3,z_4,z_5,z_6)} = F_1(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4) \\
& \overline{F_1(z_1,z_2,z_3,z_4,z_5,z_6)} = F_2(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4) \\
& \overline{F_2(z_1,z_2,z_3,z_4,z_5,z_6)} = F_3(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4) \\
& \overline{F_5(z_1,z_2,z_3,z_4,z_5,z_6)} = F_4(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4) \\
& \overline{F_6(z_1,z_2,z_3,z_4,z_5,z_6)} = F_5(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4) \\
& \overline{F_4(z_1,z_2,z_3,z_4,z_5,z_6)} = F_6(\bar z_3,\bar z_1,\bar z_2,\bar z_5,\bar z_6,\bar z_4); \\
s: \quad & F_{i+3}(z_1,z_2,z_3,z_4,z_5,z_6) = F_i(z_4,z_5,z_6,z_1,z_2,z_3) \quad i = 1,2,3, \\
& F_i(z_1,z_2,z_3,z_4,z_5,z_6) = F_{i+3}(z_4,z_5,z_6,z_1,z_2,z_3) \quad i = 1,2,3; \\
p_1: \quad & \omega_{1i}F_i(z_1,\ldots,z_6) = F_i(\omega_{11}z_1,\ldots,\omega_{16}z_6) \quad i = 1,\ldots,6; \\
p_2: \quad & \omega_{2i}F_i(z_1,\ldots,z_6) = F_i(\omega_{21}z_1,\ldots,\omega_{26}z_6) \quad i = 1,\ldots,6,
\end{aligned}$$

$$\begin{aligned}
(\omega_{11},\ldots,\omega_{16}) &= (\omega^k,\omega^\ell,\omega^{-k-\ell},\omega^k,\omega^\ell,\omega^{-k-\ell}) \\
(\omega_{21},\ldots,\omega_{26}) &= (\omega^\ell,\omega^{-k-\ell},\omega^k,\omega^{-k-\ell},\omega^k,\omega^\ell)
\end{aligned}$$

Bifurcation Equations for $M = 12$ (2)

Conditions connecting F_1 to $(F_2, \dots, F_6)$:

$$
\begin{aligned}
F_2(z_1, z_2, z_3, z_4, z_5, z_6) &= F_1(z_2, z_3, z_1, z_6, z_4, z_5) \\
F_3(z_1, z_2, z_3, z_4, z_5, z_6) &= F_1(z_3, z_1, z_2, z_5, z_6, z_4) \\
F_4(z_1, z_2, z_3, z_4, z_5, z_6) &= F_1(z_4, z_5, z_6, z_1, z_2, z_3) \\
F_5(z_1, z_2, z_3, z_4, z_5, z_6) &= F_1(z_5, z_6, z_4, z_3, z_1, z_2) \\
F_6(z_1, z_2, z_3, z_4, z_5, z_6) &= F_1(z_6, z_4, z_5, z_2, z_3, z_1)
\end{aligned}
$$

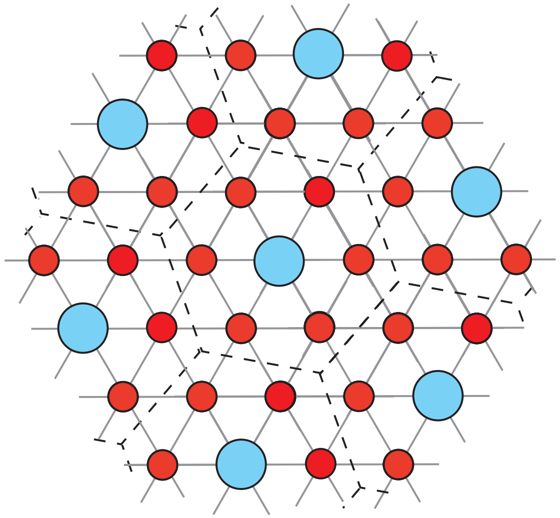
Conditions on F_1:

$$
\begin{aligned}
F_1(z_1, z_2, \dots, z_6) &= \overline{F_1(\overline{z}_1, \overline{z}_2, \dots, \overline{z}_6)} \\
\omega_{11} F_1(z_1, z_2, \dots, z_6) &= F_1(\omega_{11} z_1, \omega_{12} z_2, \dots, \omega_{16} z_6) \\
\omega_{21} F_1(z_1, z_2, \dots, z_6) &= F_1(\omega_{21} z_1, \omega_{22} z_2, \dots, \omega_{26} z_6)
\end{aligned}
$$

$$
\begin{aligned}
(\omega_{11}, \dots, \omega_{16}) = (\omega^k, \omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^\ell, \omega^{-k-\ell}) \\
(\omega_{21}, \dots, \omega_{26}) = (\omega^\ell, \omega^{-k-\ell}, \omega^k, \omega^{-k-\ell}, \omega^k, \omega^\ell)
\end{aligned}
$$

Bifurcation Equations for $M = 12$ (3)

$k = 7$



$\langle r, p_1^3 p_2, p_1^{-1} p_2^2 \rangle$ = symmetry of $(x, x, x, 0, 0, 0)$

Targeted solution: $(z_1, z_2, z_3, z_4, z_5, z_6) = (x, x, x, 0, 0, 0)$

Bifur. eqn: $F_i(z_1, z_2, z_3, z_4, z_5, z_6) = 0 \quad (i = 1, \ldots, 6)$

$\Longleftrightarrow \quad F_1(x, x, x, 0, 0, 0) = 0, \quad F_1(0, 0, 0, x, x, x) = 0$

Bifurcation Equations for $M = 12$ (4)

For $(k, \ell, n) = (2, 1, 7)$:

$$
\begin{aligned}
F_1 = \; & A_1 z_1 + (A_2 \bar{z}_2 \bar{z}_3 + A_3 \bar{z}_1 z_3 + A_4 z_2^2) \\
& + (A_5 z_1^2 \bar{z}_1 + A_6 z_1 z_2 \bar{z}_2 + A_7 z_1 z_3 \bar{z}_3 + A_8 z_1 z_4 \bar{z}_4 \\
& \quad + A_9 z_1 z_5 \bar{z}_5 + A_{10} z_1 z_6 \bar{z}_6 + A_{11} \bar{z}_1 z_2 \bar{z}_3 + A_{12} z_2 z_3^2 \\
& \quad + A_{13} \bar{z}_2^2 z_3 + A_{14} \bar{z}_1^2 \bar{z}_2 + A_{15} \bar{z}_3^3) + \cdots
\end{aligned}
$$

$\Downarrow$

$$
\begin{aligned}
F_1(x, x, x, 0, 0, 0) &= A_1 x + (A_2 + A_3 + A_4) x^2 \\
& \quad + (A_5 + A_6 + \cdots + A_{15}) x^3 + \cdots \\
&\approx x(A\tilde{\tau} + Bx) \\
F_1(0, 0, 0, x, x, x) &= 0
\end{aligned}
$$

$$\Longrightarrow x \approx -(A/B)\tilde{\tau}$$

Bifurcation Equations for $M = 12$ (5)

For $(k, \ell, n) = (2, 1, 6)$:

$$\begin{aligned} F_1 \;=\; & A_1 z_1 + A_2 \overline{z}_2 \overline{z}_3 + (A_3 z_1^2 \overline{z}_1 + A_4 z_1 z_2 \overline{z}_2 + A_5 z_1 z_3 \overline{z}_3 \\ & + A_6 z_1 z_4 \overline{z}_4 + A_7 z_1 z_5 \overline{z}_5 + A_8 z_1 z_6 \overline{z}_6 + A_9 z_2 \overline{z}_4 z_6 \\ & + A_{10} z_3 \overline{z}_4 z_5 + A_{11} \overline{z}_1 z_2 \overline{z}_6 + A_{12} z_3^2 z_4 + A_{13} \overline{z}_1 \overline{z}_5^2) \\ & + \; [A_{14} z_4 \overline{z}_6^2 + A_{15} \overline{z}_5 z_6^3 + A_{16} \overline{z}_5 \overline{z}_6^3 + \cdots] \; + \cdots \end{aligned}$$

$\Downarrow$

$$F_1(x, x, x, 0, 0, 0) = A_1 x + A_2 x^2 + (A_3 + A_4 + A_5) x^3 + \cdots$$

$$F_1(0, 0, 0, x, x, x) = A_{14} x^3 + (A_{15} + A_{16}) x^4 + \cdots$$

Two equations in one variable x

$\Longrightarrow$ No solution exists

Bifurcation at 12-fold Critical Point

Irred rep: $(12; k, \ell)$

	$\gcd(\hat{k}-\hat{\ell},\hat{n}) \notin 3\mathbb{Z}$	$\gcd(\hat{k}-\hat{\ell},\hat{n}) \in 3\mathbb{Z}$
	$\hat{D} \notin 3\mathbb{Z}$	$\hat{D} \in 3\mathbb{Z}$
GCD-div	traffic-like (type V)	market-like (type M)
$\overline{\text{GCD-div}}$	traffic-like (V) admin-like (T)	market-like (M) admin-like (T)

$$\hat{k} = \frac{k}{\gcd(k,\ell,n)}, \quad \hat{\ell} = \frac{\ell}{\gcd(k,\ell,n)}, \quad \hat{n} = \frac{n}{\gcd(k,\ell,n)}$$

GCD-div:

$(\hat{k}-\hat{\ell})\gcd(\hat{k},\hat{\ell})$ is divisible by $\gcd(\hat{k}^2+\hat{k}\hat{\ell}+\hat{\ell}^2,\hat{n})$

Summary of Our Results (again)

Christaller's	size n	Mult M
$k = 3$ (market)	$3\times$	2
$k = 4$ (traffic)	$2\times$	3
$k = 7$ (administrative)	$7\times$	12

Lösch's D	size n	Mult M
9 (traffic-like)	$3\times$	6
12 (market-like)	$6\times$	6
13 (admin-like)	$13\times$	12
16 (traffic-like)	$4\times$	6
19 (admin-like)	$19\times$	12
21 (admin-like)	$21\times$	12
25 (traffic-like)	$5\times$	6

Group

Appendix

(i) **Associative law:** $(g\ h)\ k = g\ (h\ k)$

(ii) $\exists$ **identity element** e**:** $e\ g = g\ e = g\ (\forall g \in G)$

(iii) $\forall g \in G$, $\exists h$ **(inverse of** g**):** $g\ h = h\ g = e$

Dihedral group D_6

$$\mathrm{D}_6 = \langle r, s\rangle \ = \ \{e, r, r^2, \ldots, r^5, s, sr, sr^2, \ldots, sr^5\}$$

$$r^6 = s^2 = (sr)^2 = e$$

Semidirect product $G = \mathrm{D}_6 \ltimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

- $\mathbb{Z}_n \times \mathbb{Z}_n$ **is a normal subgroup of** G
- **unique representation** $g = ha$ $(h \in \mathrm{D}_6,\ a \in \mathbb{Z}_n \times \mathbb{Z}_n)$

Equivariant Branching Lemma

Appendix

Assume that rep $\tilde{T}$ is absolutely irreducible and the bifurcation equation is "generic."

For an isotropy subgroup Σ with $\dim \mathrm{Fix}(\Sigma) = 1$, there exists a unique smooth solution branch s.t.

$\Sigma(w) = \Sigma$ for each solution w on the branch.

$$\begin{aligned} \Sigma(w) &= \{g \in G \mid \widetilde{T}(g)w = w\} \\ \mathrm{Fix}(\Sigma) &= \{w \mid \widetilde{T}(g)w = w \text{ for all } g \in \Sigma\} \end{aligned}$$