Hausdorff Trimester Program: Combinatorial Optimization
Rigidity Workshop, Bonn, October 5-9, 2015

Extensions and Ramifications of Discrete Convexity Concepts

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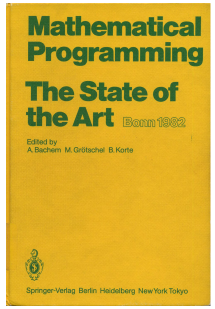
151006BonnHIM

Submodular functions and convexity



11th Math.Prog.Symp, Bonn, 1982

L. Lovász
Eötvös Loránd University, Department of Analysis I, Múzeum krt. 6–8, H-1088 Budapest, Hungary

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– Convex functions occur in many mathematical models in economy, engineering, and other sciencies. Convexity is a very natural property of various functions and domains occuring in such models; quite often the only non-trivial property which can be stated in general.

– Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.

– Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.

– There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.

Features of Convex Functions

- Occurrence in many models

  motivations, applications

- Operations and transformations

- Sufficient structure for a theory

  mathematically beautiful, practically useful

- Minimization algorithms

Features of Convex Functions

and submodular functions

- Occurrence in many models

  motivations, applications

- Operations and transformations

- Sufficient structure for a theory

  mathematically beautiful, practically useful

- Minimization algorithms

Features of Convex Functions

= Issues in discrete convex analysis

- Occurrence in many models ?

  motivations, applications

- Operations and transformations ?

- Sufficient structure for a theory ?

  mathematically beautiful, practically useful

- Minimization algorithms ?

THIS TALK	defi-nition	model appl.	opera-tions	proper-ties
submodular				
separable-convex				
integrally-convex				
L-convex ($\mathbb{Z}^n$)				
M-convex ($\mathbb{Z}^n$)				
M-convex (jump)				
L-convex (graph)				

Some History

1935 Matroid — Whitney, Nakasawa

1965 Submodular function — Edmonds

1969 Convex network flow (electr.circuit) — Iri

1982 Submodularity and convexity — Frank, Fujishige, Lovász

1990 Valuated matroid — Dress–Wenzel
Integrally convex fn — Favati–Tardella

1996 Discrete convex analysis — Murota

2000 Submodular minimization algorithm — Iwata–Fleischer–Fujishige, Schrijver

2006 M-convex fn on jump system — Murota

2012 L-convex fn on graph — Hirai, Kolmogorov

Classes of
Discrete Convex Functions

Classes of Discrete Convex Functions

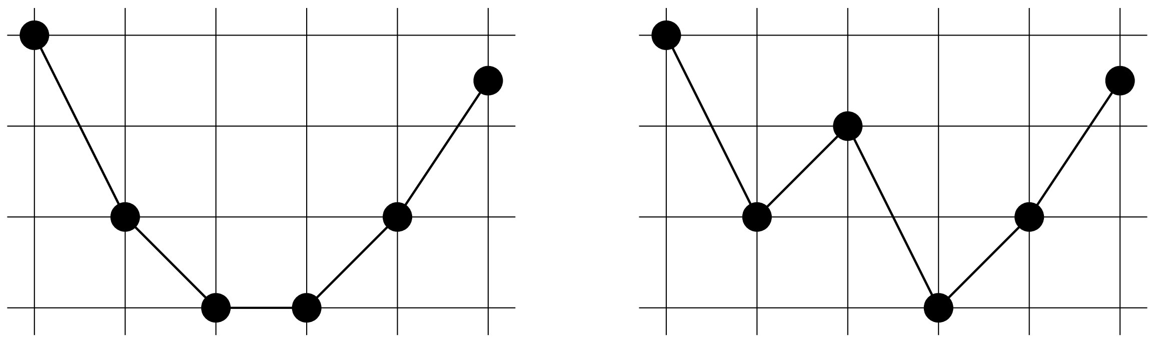
1. Submodular set fn (on $\{0,1\}^n$)
1. Separable-convex fn on $\mathbb{Z}^n$
1. Integrally-convex fn on $\mathbb{Z}^n$

2. L-convex (L$^\natural$-convex) fn on $\mathbb{Z}^n$
2. M-convex (M$^\natural$-convex) fn on $\mathbb{Z}^n$

3. M-convex fn on jump systems
3. L-convex fn on graphs

Definitions

submodular (set fn)	$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$
separable-conv	$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$ $\varphi_i(t-1) + \varphi_i(t+1) \geq 2\varphi_i(t) \ \ (\forall t \in \mathbf{Z})$
integrally-conv	
L-conv ($\mathbb{Z}^n$)	
M-conv ($\mathbb{Z}^n$)	
M-conv (jump)	
L-conv (graph)	

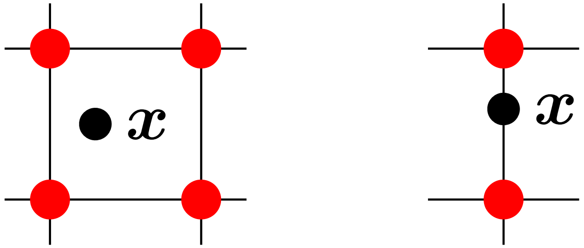


convex non-convex

Integrally Convex Function

(Favati-Tardella 1990)

$$N(x) = \{y \in \mathrm{Z}^n \mid \|x - y\|_\infty < 1\} \qquad (x \in \mathrm{R}^n)$$



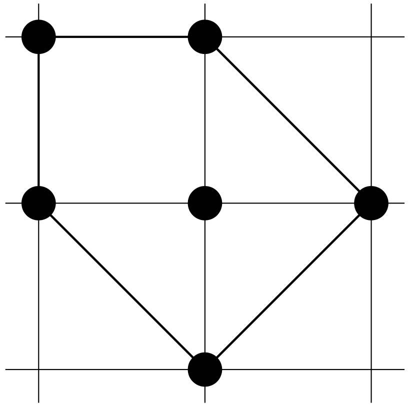
Local convex extension:

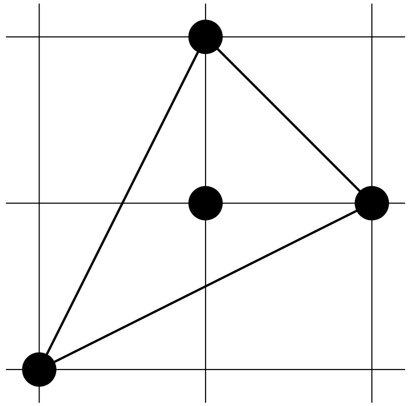
$$\tilde{f}(x) = \sup_{p,\alpha}\{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \le f(y) \ \ (\forall y \in N(x))\}$$

Def: f is integrally convex $\iff$ $\tilde{f}$ is convex

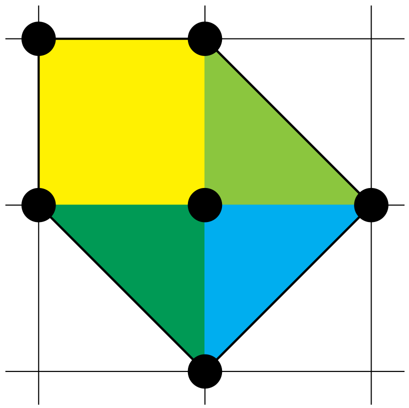
Rem: Every set function is integrally convex

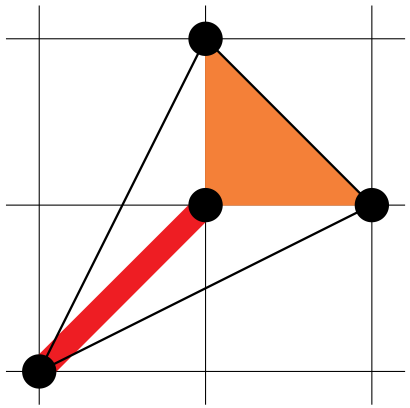
Integrally Convex Set



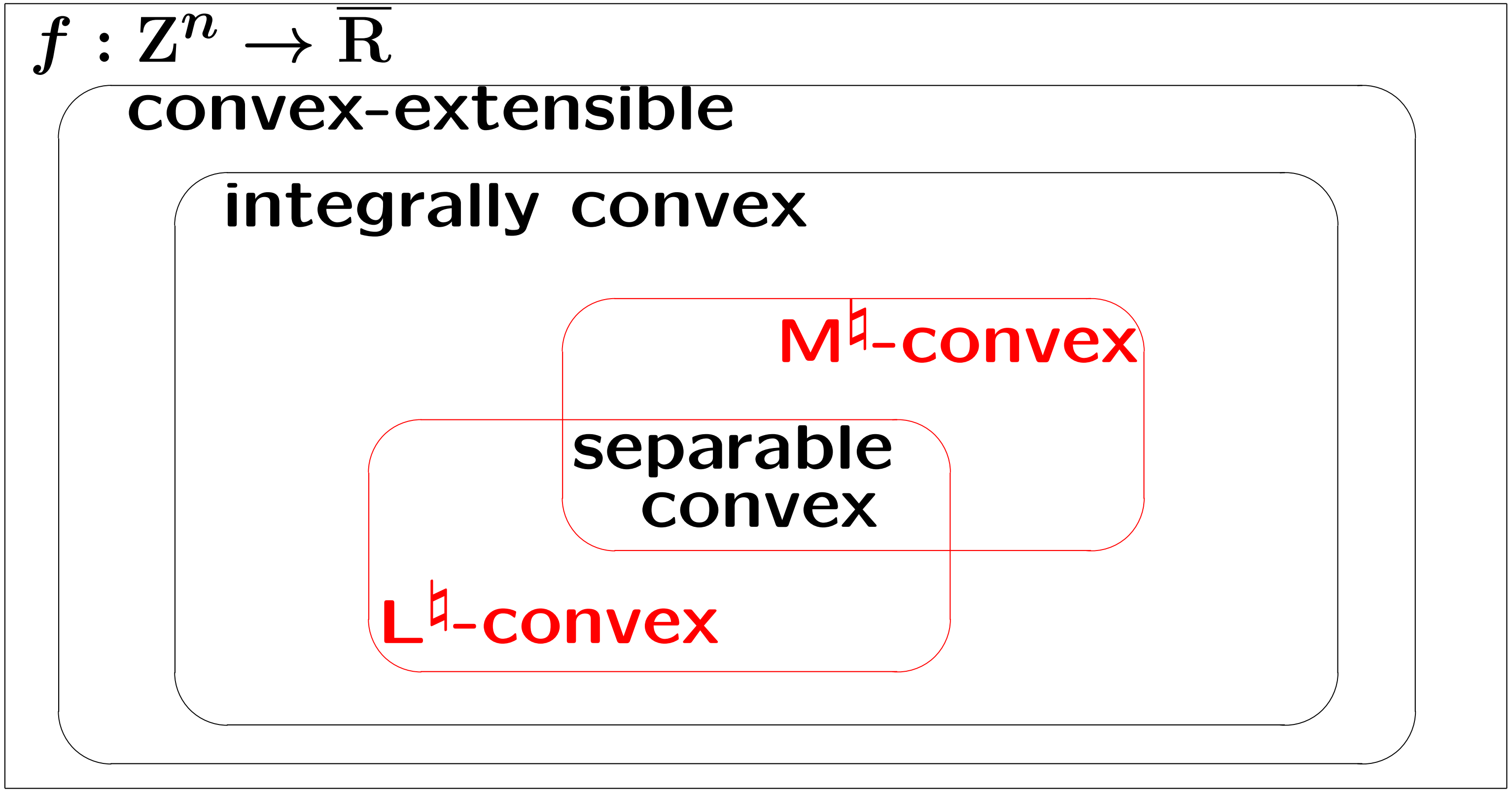


YES NO





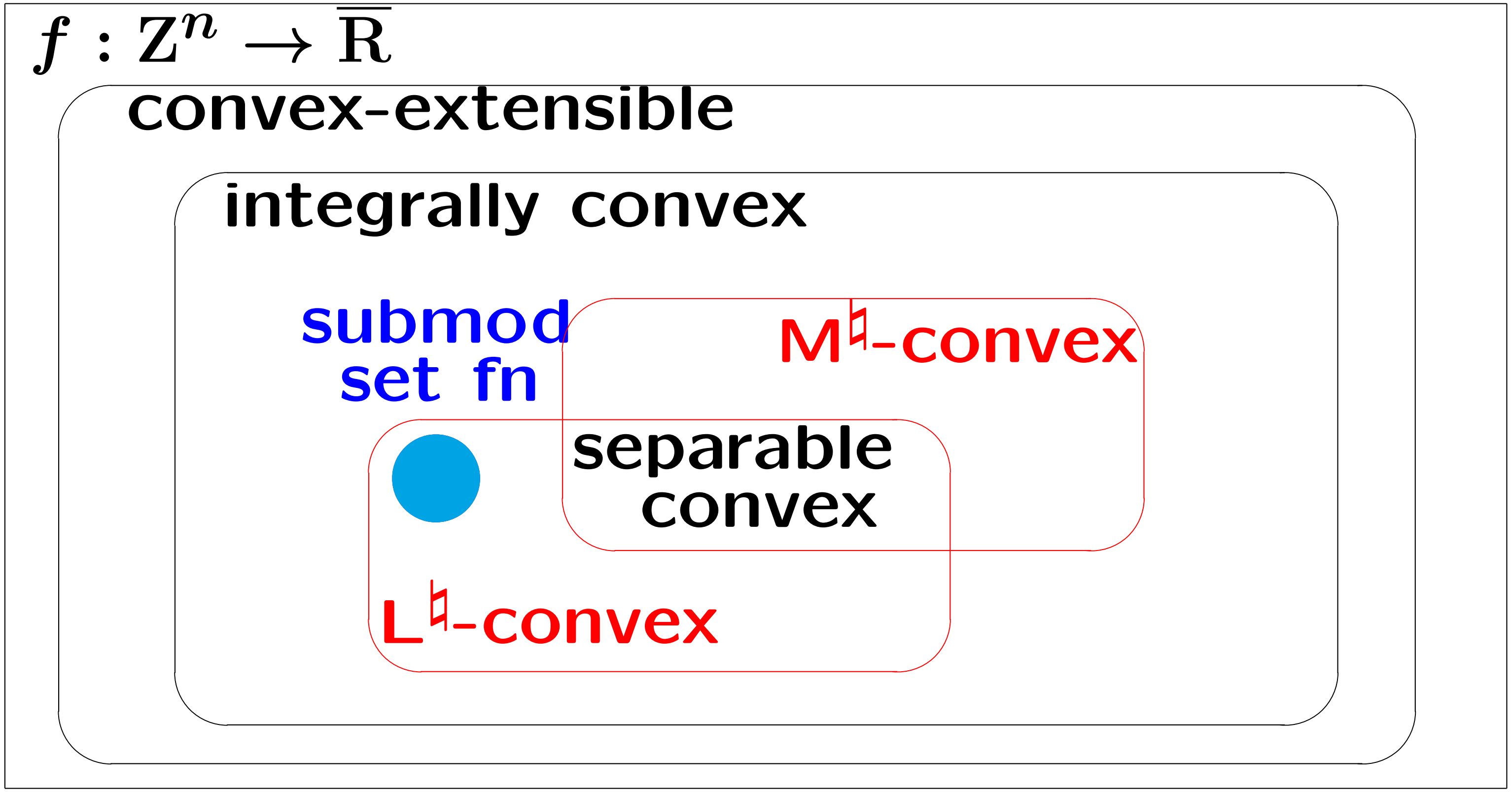
Classes of Discrete Convex Functions



f: convex-extensible

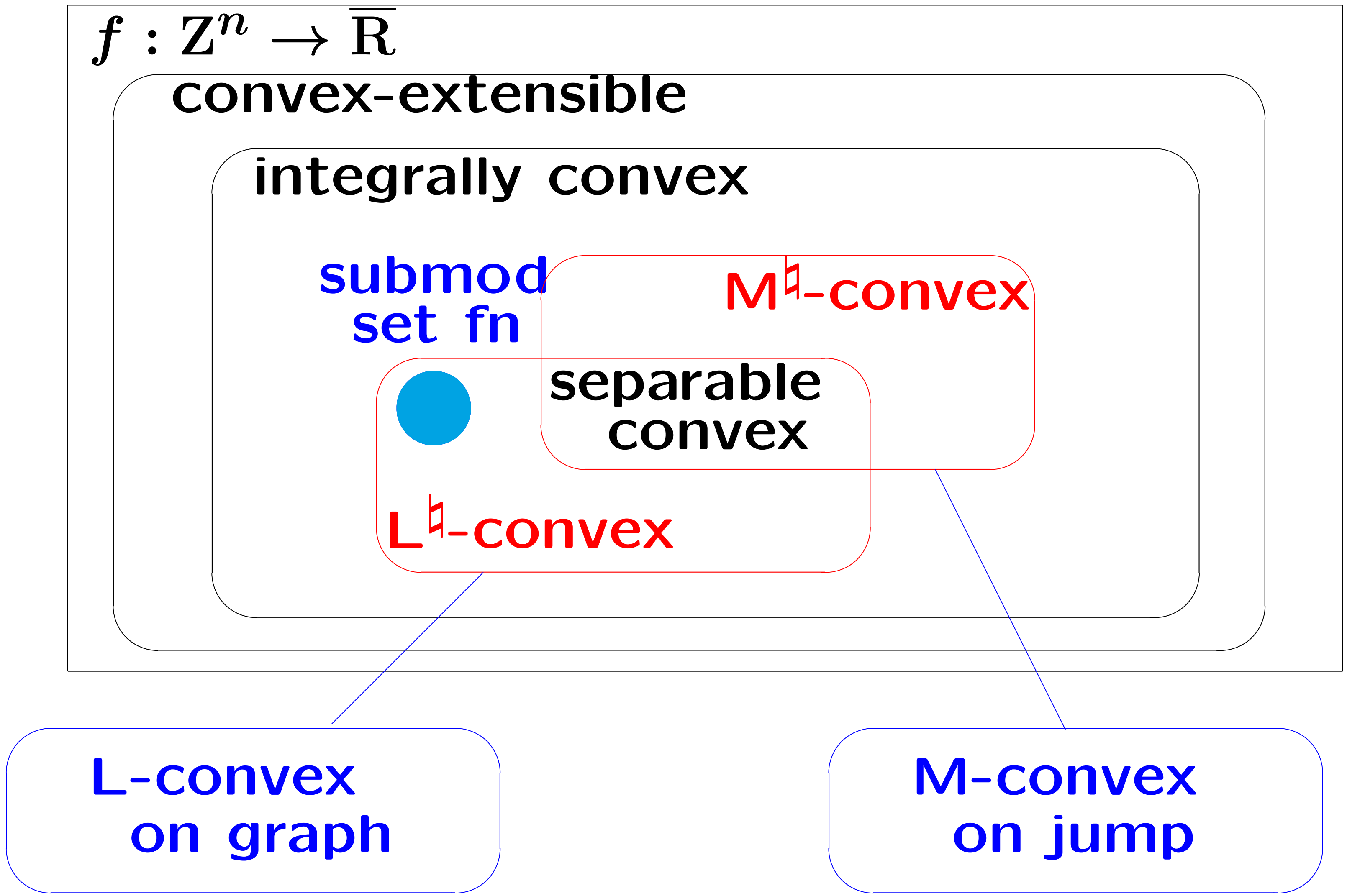
$\Leftrightarrow$ $\exists$ convex $\overline{f}$: $f(x) = \overline{f}(x)$ $(\forall x \in \mathbf{Z}^n)$

Classes of Discrete Convex Functions



integrally convex $\cap$ submodular $=$ L♮-convex

Classes of Discrete Convex Functions



multiflow, tight span
metric graph theory
valued CSP

matching theory

Motivations/Applications/Connections

1. submodular	MANY problems connectivity (crossing), rigidity
1. separable-conv	MANY problems min-cost flow, resource allocation
1. integrally-conv	[mathematical aesthetics]
2. L-conv ($\mathbb{Z}^n$)	network tension, image processing OR (inventory, scheduling)
2. M-conv ($\mathbb{Z}^n$)	network flow, conjestion game economics (game, auction) mixed polynomial matrix
3. M-conv (jump)	deg sequence, (2-)matching polynomial (half-plane property)
3. L-conv (graph)	multiflow, multifacility location

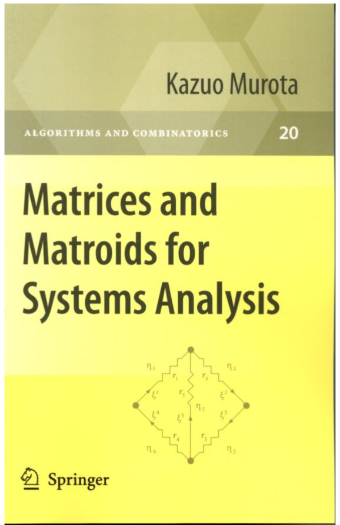
Books (discrete convex analysis)

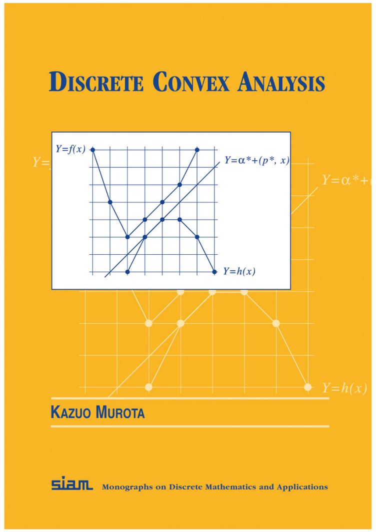
2000: Murota, Matrices and Matroids for Systems Analysis, Springer

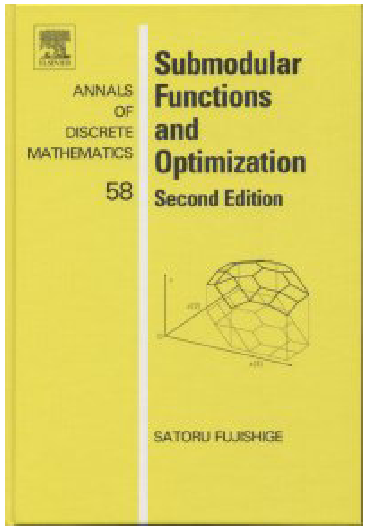
2003: Murota, Discrete Convex Analysis, SIAM

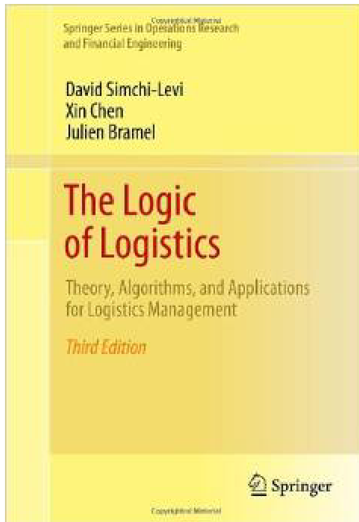
2005: Fujishige, Submodular Functions and Optimization, 2nd ed., Elsevier

2014: Simchi-Levi, Chen, Bramel, The Logic of Logistics, 3rd ed., Springer









Definitions

1. submodular (set fn)	$\rho(X)+\rho(Y)\geq\rho(X\cup Y)+\rho(X\cap Y)$
1. separable -conv	$f(x)=\varphi_1(x_1)+\varphi_2(x_2)+\cdots+\varphi_n(x_n)$ $\varphi_i(t-1)+\varphi_i(t+1)\geq 2\varphi_i(t)\ \ (\forall t\in \mathrm{Z})$
1. integrally -conv	Local convex ext $\tilde{f}(x)$ is convex
2. L-conv ($\mathbb{Z}^n$)	
2. M-conv ($\mathbb{Z}^n$)	
3. M-conv (jump)	
3. L-conv (graph)	

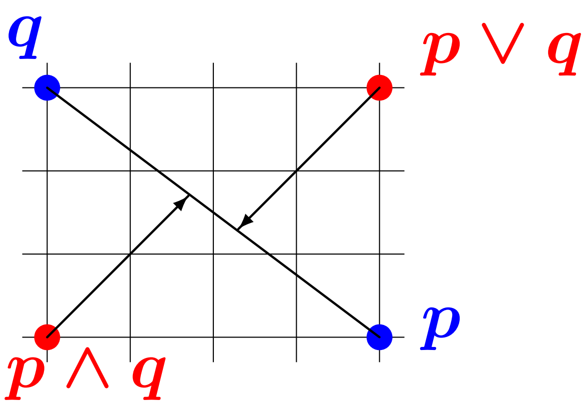
L-convex Function (L = Lattice)

(M. 98)

$g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

$p \vee q$ compnt-max

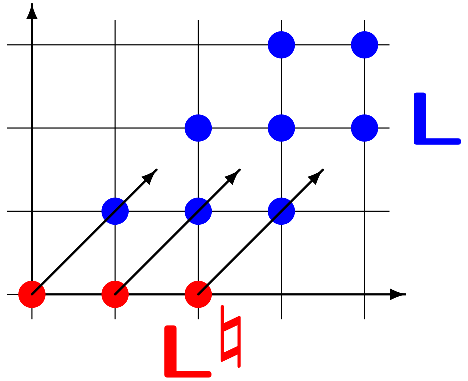
$p \wedge q$ compnt-min



Def: g is L-convex $\iff$

- Submodular: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$
- Translation: $\exists r,\ \forall p$: $g(p + \mathbf{1}) = g(p) + r$

$$\mathbf{1} = (1, 1, \ldots, 1)$$



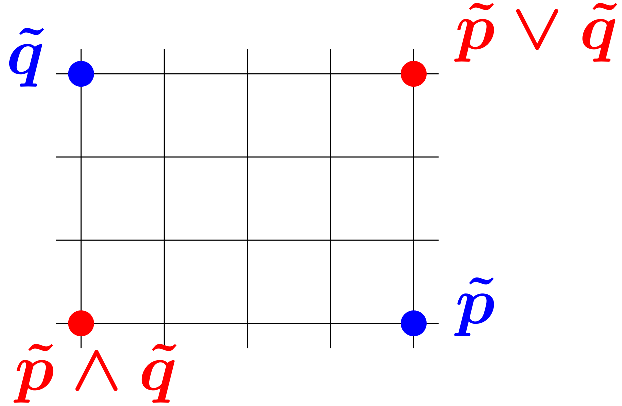
L$^\natural$-convexity from Submodularity

$g : \mathbf{Z}^n \to \mathbf{R}$ **L$^\natural$-convex** $\iff$ **(M. 98)**

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ **is submodular in** (p_0, p)

$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1)$

$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}\,(\tilde{p} \vee \tilde{q}) + \tilde{g}\,(\tilde{p} \wedge \tilde{q})$



$\mathrm{L}_{n+1} \simeq \mathrm{L}_n^\natural \supsetneqq \mathrm{L}_n$

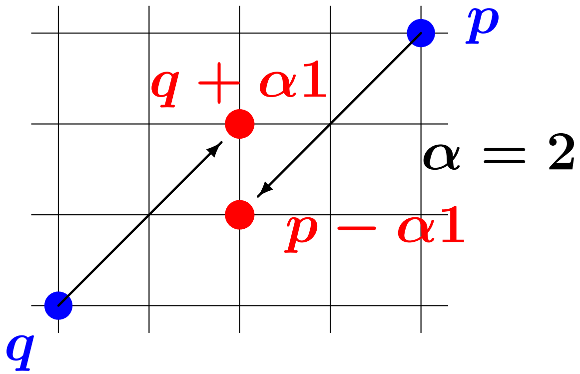
Translation Submod./ Disc Mid-point Convex

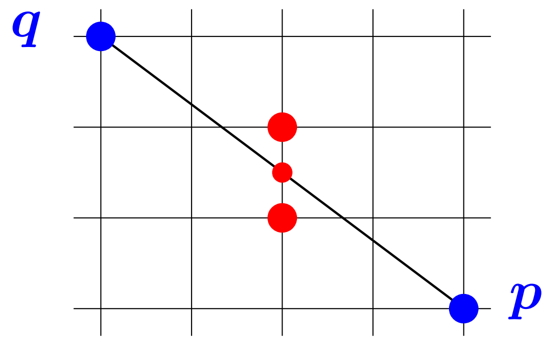
$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1)) \quad (\alpha \geq 0)$$

translation submodular (Fujishige-M. 00)

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

disc mid-pt convex (Favati-Tardella 90)





$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

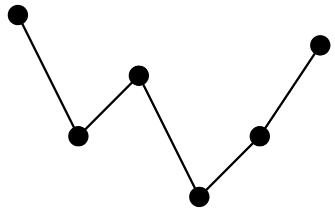
$\Leftrightarrow$ translation submodular (Fujishige-M. 00)

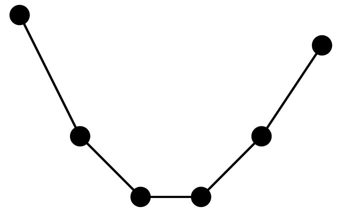
$\Leftrightarrow$ discrete mid-pt convex (Fujishige-M. 00)

$\Leftrightarrow$ submod. integ. convex (Favati-Tardella 90)

Rem: L$^\natural$-convex vs Submodular

$n = 1$





Fact 1: **Every** $g : \mathbf{Z} \to \mathbf{R}$ **is submodular**

Fact 2: **Function** $g : \mathbf{Z} \to \mathbf{R}$ **is L$^\natural$-convex**

$\iff g(p-1) + g(p+1) \geq 2g(p)$ **for all** $p \in \mathbf{Z}$

L$^\natural$-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Energy function: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathrm{R}}$

$$\Leftrightarrow \rho(X) = g(\chi_X) \quad \text{for some L}^\natural\text{-convex } g$$

Multimodular: $h : \mathrm{Z}^n \to \overline{\mathrm{R}}$ is multimodular $\Leftrightarrow$

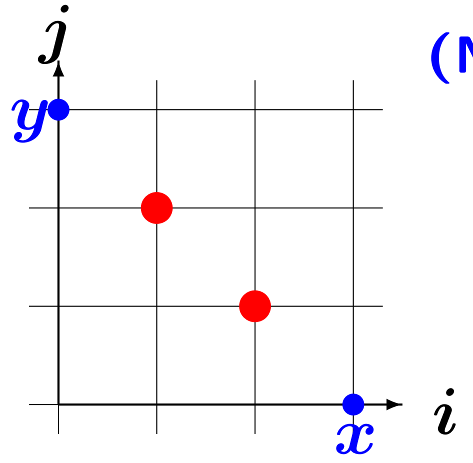
$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex g

M-convex Function

(M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector



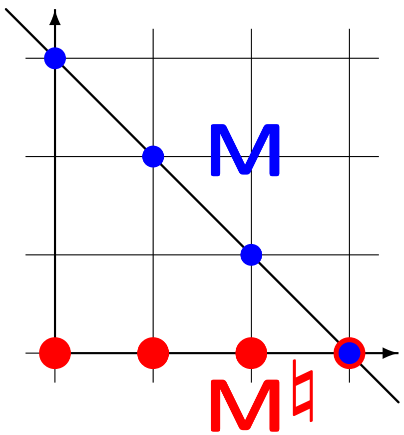
(M. 96)

Def: f is M-convex

$\iff \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

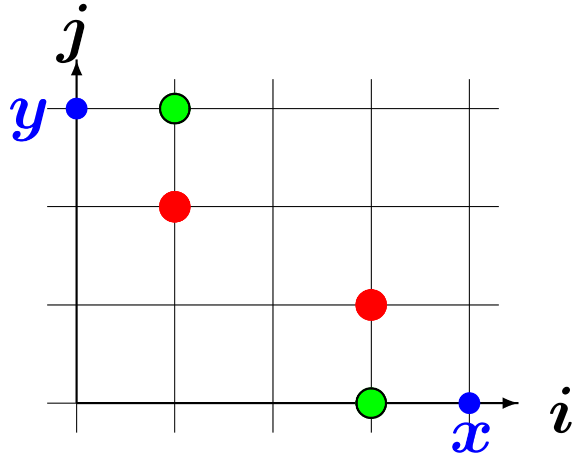
$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom}\, f \subseteq$ const-sum hyperplane



M♮-convex Function

(M. -Shioura 99)



$$f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$$

f: **M♮-convex** $\Leftrightarrow \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \right.$$

$$\left. \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

$$\mathbf{M}_{n+1} \simeq \mathbf{M}^\natural_n \supsetneq \mathbf{M}_n$$

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum_i \sum_j a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow a_{ij} \geq 0, \quad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i, j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Matroid rank: $f(X) = -$ rank of X

Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum_i \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

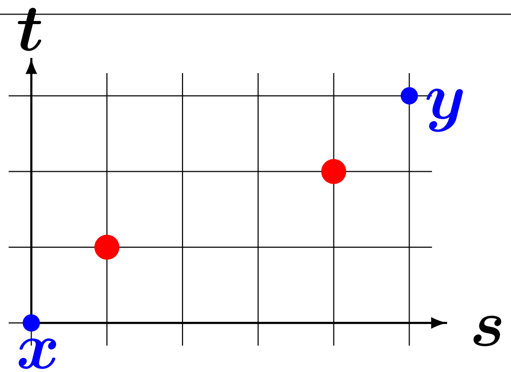
$\{A, B, \ldots.\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

Constant-Parity Jump System

J: **const-parity jump system** (Geelen)

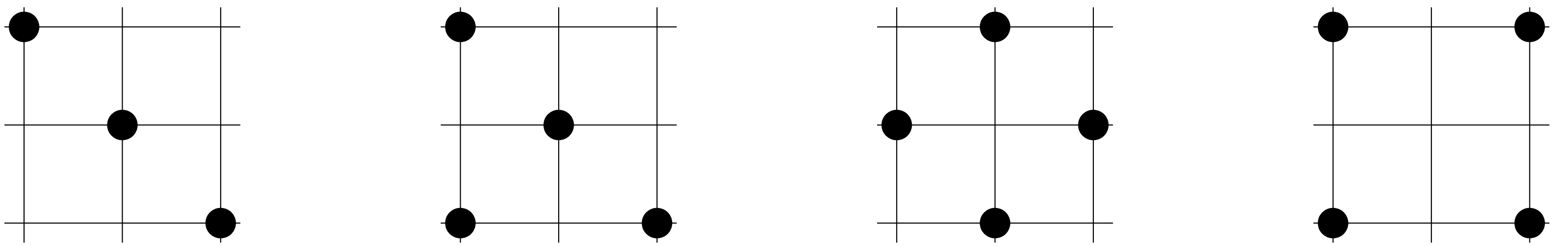
$\iff$ $\forall x, y \in J$, $\forall$ (x,y)-incr s, $\exists$ $(x+s,y)$-incr t

s.t. $x+s+t,\ y-s-t \in J$



degree sequence
matching, factor

simultaneous exchange



M-convex Function on Jump Systems

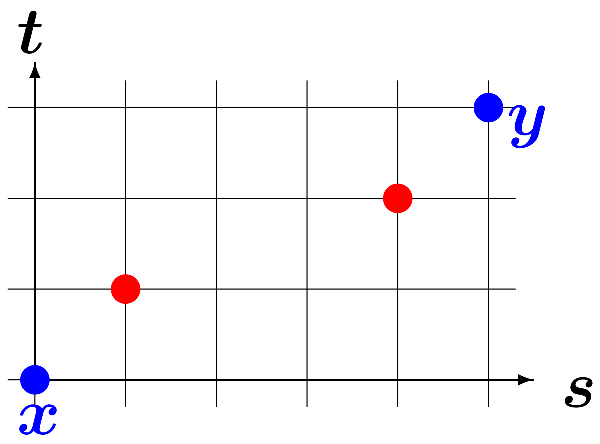
(M. 06)

J: const-parity jump system, $f : J \to \mathrm{R}$

$\forall\ x, y \in J$ and (x, y)-incr s,

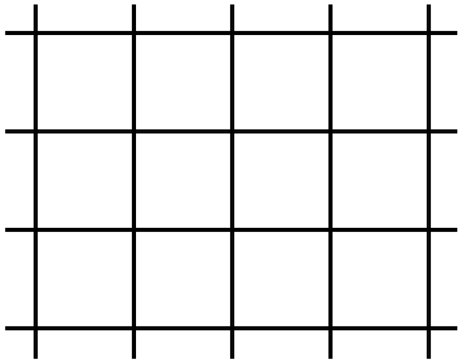
$\exists\ (x + s, y)$-incr t s.t. $x + s + t,\ y - s - t \in J$,

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t)$$

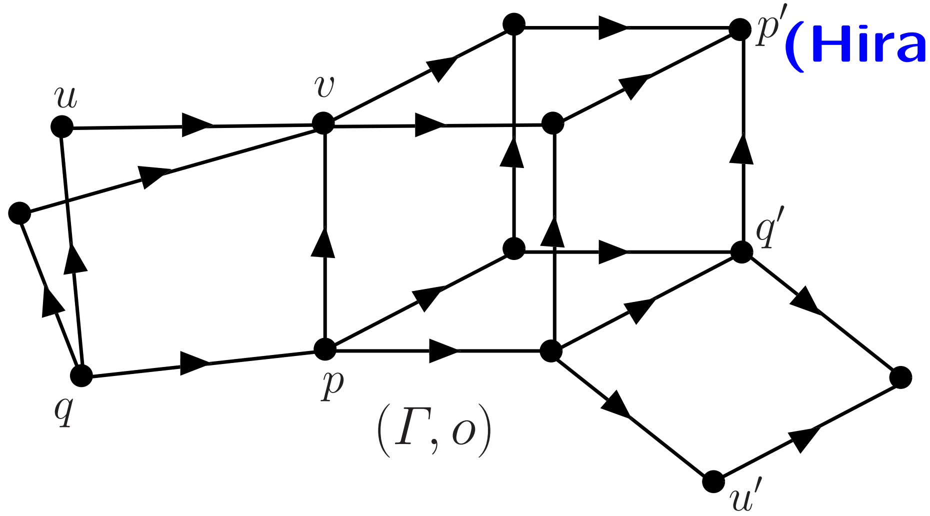


- Minsquare factor (Apollonio–Sebő 04)
- Even factor (Kobayashi–Takazawa 09)

L-convex Function on Graphs



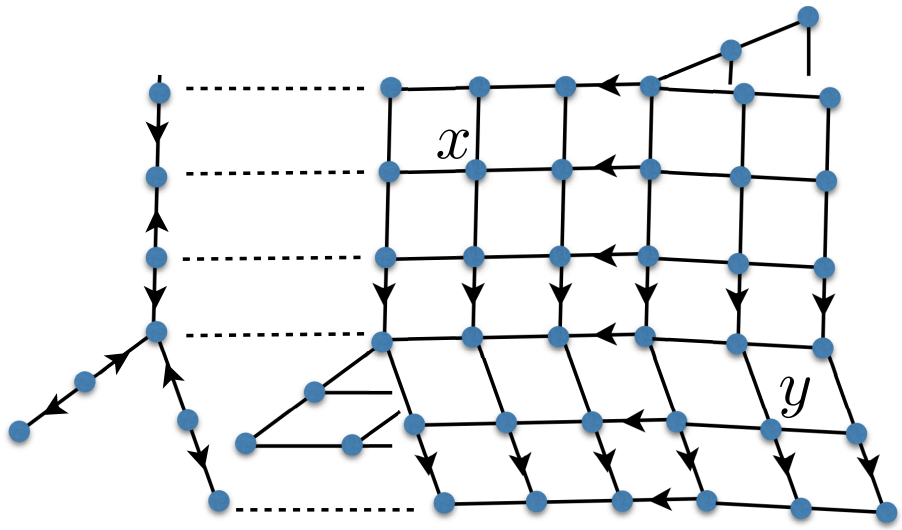
integer lattice



(Hirai 12)

oriented modular graph

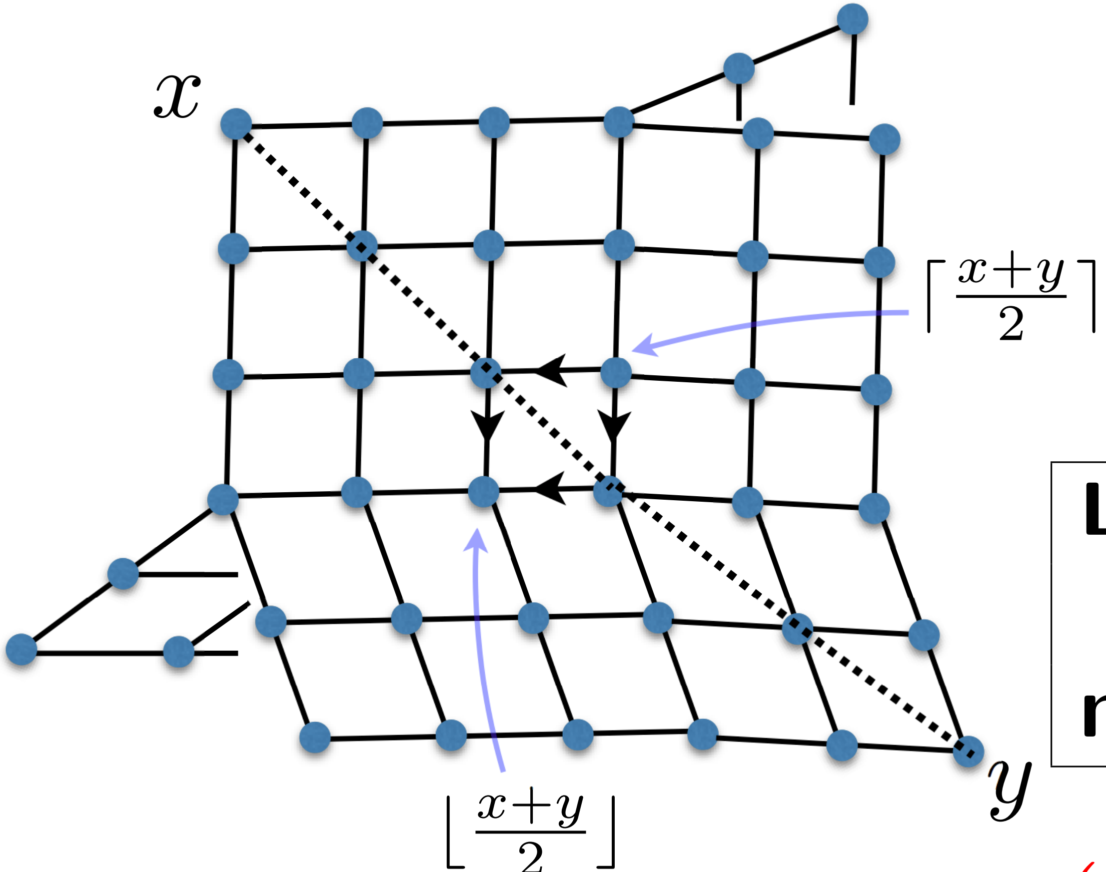
direct product of trees:



(Kolmogorov 11)
(Huber-Kolmogorov 12)

Mid-point Convexity on Tree Products

(Hirai 13,15)



L-convex
|| (def)
mid-point convex

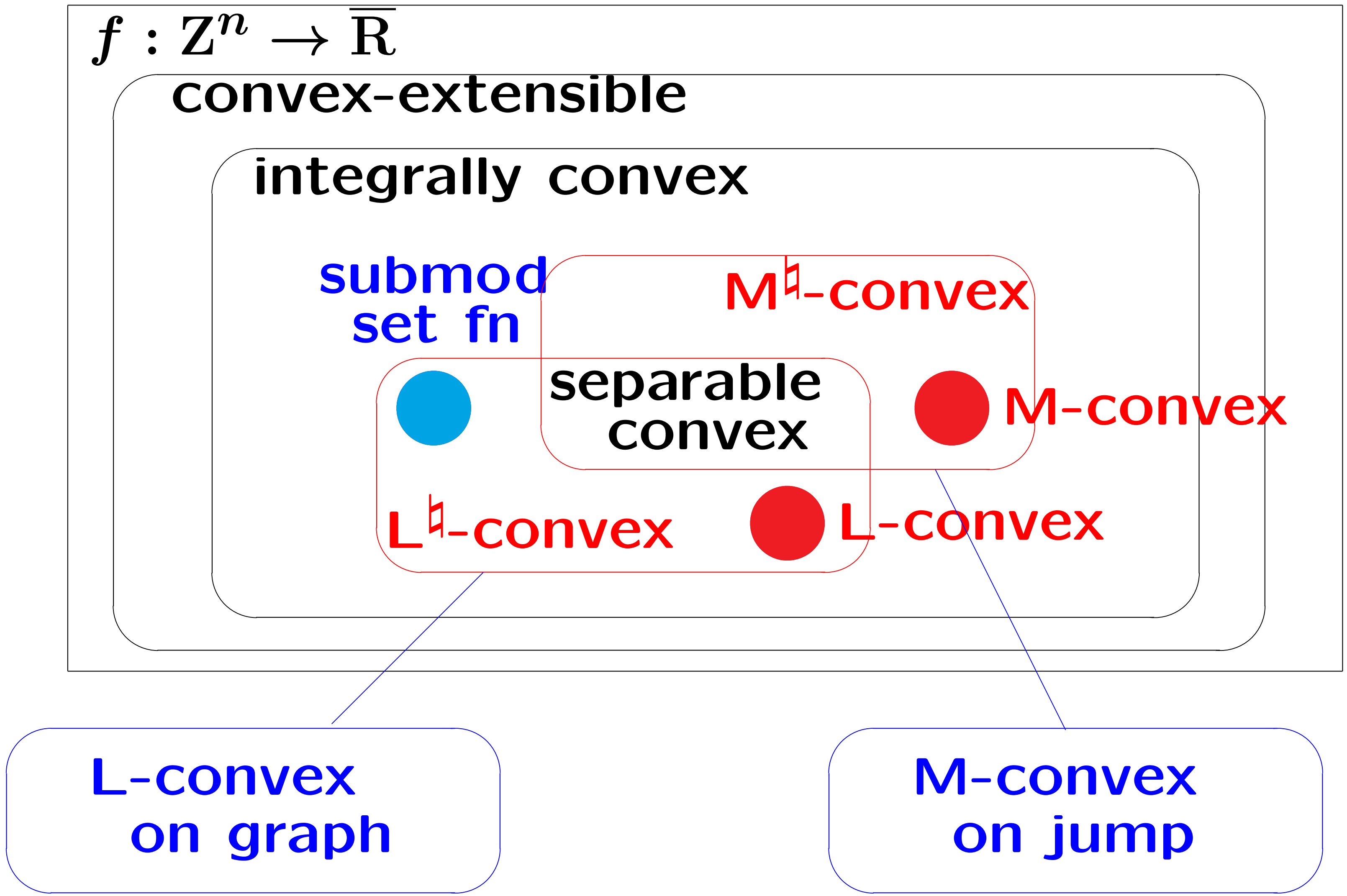
$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$$

- submodular on (rooted) trees (Kolmogorov 11)
- k-submodular (Huber-Kolmogorov 12)

Definitions

1. submodular (set fn)	$\rho(X)+\rho(Y) \geq \rho(X \cup Y)+\rho(X \cap Y)$
1. separable -conv	$f(x)=\varphi_1(x_1)+\varphi_2(x_2)+\cdots+\varphi_n(x_n)$
1. integrally -conv	Local convex ext $\tilde{f}(x)$ is convex
2. L-conv ($\mathbb{Z}^n$) ($\mathrm{L}^\natural$)	$g(p)+g(q)$ $\geq g((p-\alpha 1) \vee q)+g(p \wedge (q+\alpha 1))$ $\Leftrightarrow\ g(p)+g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right)+g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$
2. M-conv ($\mathbb{Z}^n$)	$f(x)+f(y)$ $\geq f(x-e_i+e_j)+f(y+e_i-e_j)$
3. M-conv (jump)	$f(x)+f(y)$ $\geq f(x \pm e_i \pm e_j)+f(y \mp e_i \mp e_j)$
3. L-conv (tree^n)	$g(p)+g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right)+g\left(\left\lceil \frac{p+q}{2} \right\rfloor\right)$

Classes of Discrete Convex Functions



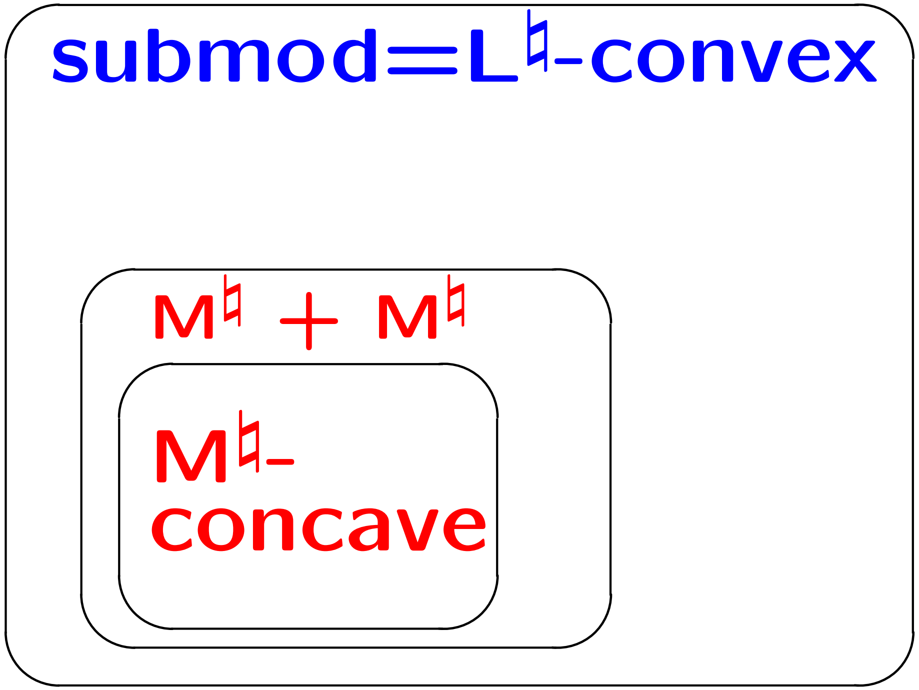
integrally convex $\cap$ submodular = L$^\natural$-convex

Submodular Set Function in DCA

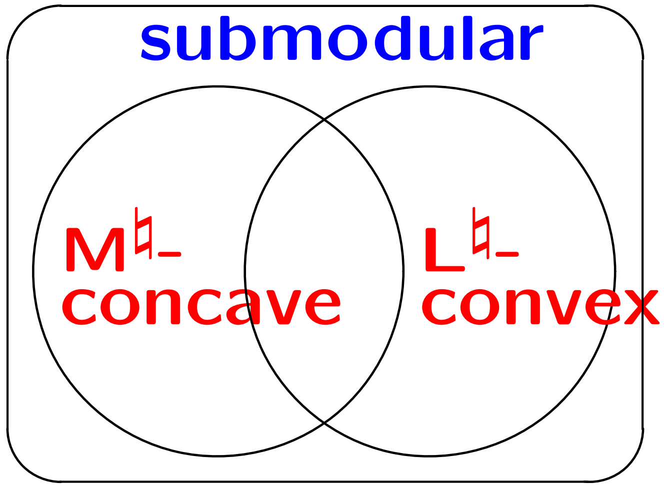
- Every set func is integrally convex
- Submodular set func = L$^\natural$-convex on $\{0,1\}^n$
- (Sums of) M$^\natural$-concave form nice subclasses

Max. submod f s.t. matroid

$f: \{0,1\}^n \to \mathrm{R}$



$f: \mathrm{Z}^n \to \mathrm{R}$



Operations

Operations

- **scaling:** $af(x)+b$, $\quad f(ax+b)$, $\quad f(Ax+b)$
- **linear addition:** $f(x)+\langle p,x\rangle$
- **section:** $f(x,0)$
- **projection:** $\min_y f(x,y)$
- **sum:** $f_1(x)+f_2(x)$
- **convolution:** $(f_1 \Box f_2)(x)=\min_y (f_1(y)+f_2(x-y))$

◇ **transformation by graphs/networks**

Scaling/Linear Addition

	$af(x)$ $(a > 0)$	$f(sx)$ $(s \in Z_+)$	$f(-x)$	$f(x) + \langle p, x \rangle$
submodular (set fn)	Y	—	Y* $V \setminus X$	Y
separable-conv	Y	Y	Y $\pm x_i$	Y
integrally-conv	Y	Y* $(n = 2)$	Y $\pm x_i$	Y
L-conv $(\mathbb{Z}^n)$	Y	Y	Y	Y
M-conv $(\mathbb{Z}^n)$	Y	N	Y	Y
M-conv (jump)	Y	N	Y $\pm x_i$	Y
L-conv (star^n)	Y	?	—	Y* dist(0,·)

Section/Projection

	section $f(x,0)$	projection $\min_y f(x,y)$
submodular (set fn)	Y restriction	Y contraction
separable-conv	Y	Y
integrally-conv	Y	?
L-conv ($\mathbb{Z}^n$)	N	Y
$\mathrm{L}^\natural$-conv	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	N
$\mathrm{M}^\natural$-conv	Y	Y
M-conv (jump)	Y	Y
L-conv (tree^n)	Y	Y

Sum/Convolution

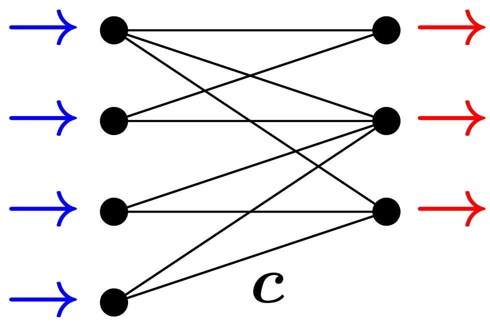
	sum $f_1 + f_2$	convolution $f_1 \,\square\, f_2$
submodular (set fn)	Y	N matroid intersec $\min_{Y \subseteq X} (\rho_1(Y) + \rho_2(X \setminus Y))$
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	N $\to$ L_2-convex
M-conv ($\mathbb{Z}^n$)	N $\to$ M_2-conv matr.intersec	Y matroid union
M-conv (jump)	N	Y
L-conv (tree^n)	Y	—

Transformation by Graph/Network

Given
G, f, c

$f(\cdot)$

$\boldsymbol{x}$



$\boldsymbol{y}$

$g(\cdot)$

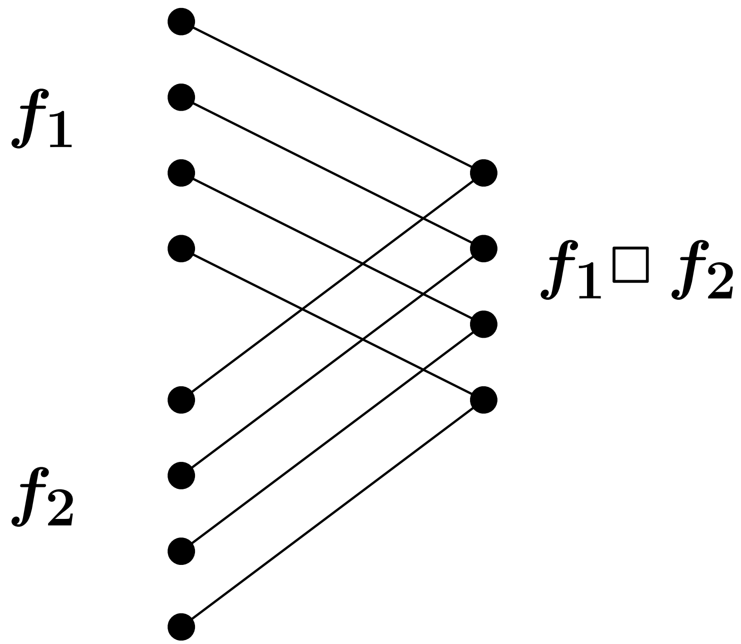
$$\min\{f(\boldsymbol{x}) + c(\xi) \mid \boldsymbol{x} \longleftrightarrow \boldsymbol{y}\} =: g(\boldsymbol{y})$$

$\uparrow$

$\exists$ **flow** $\xi : \partial\xi = x \oplus (-y)$

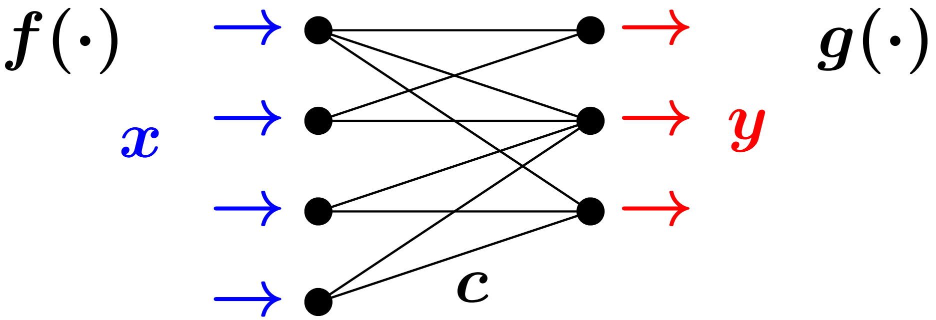
$\exists$ **matching** $M : \partial M = X \cup Y$

Rem: Convolution



$$\min_y (f_1(y) + f_2(x - y)) = (f_1 \Box f_2)(x)$$

Transformation by Graph/Network



$$\min\{f(x) + c(\xi) \mid x \longleftrightarrow y\} =: g(y)$$

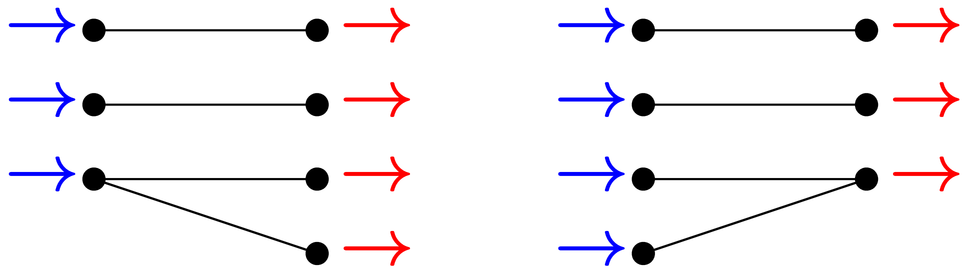
Theorem: (1) M. 96, (2) Kobayashi-M.-Tanaka 07

(1) f : M-convex ($\mathbb{Z}^n$) $\Longrightarrow$ g: M-convex ($\mathbb{Z}^n$)

(2) f : M-convex (jump) $\Longrightarrow$ g: M-convex (jump)

Proof:

elementary cases:



Open Question (posed by A. Frank)

$G = (V, W; E)$: bipartite graph*, $c : E \to \mathrm{R}$: edge cost, $(V, \mathcal{I})$: matroid ($\mathcal{I}$: indep. sets). Then

$$g(Y) = \min\{c(M) \mid M\text{: matching, } V \cap \partial M \in \mathcal{I}, \quad W \cap \partial M = Y\} \quad (Y \subseteq W)$$

is known to be an M♮-convex set function†.

Question: Does every M♮-convex set function $g : 2^W \to \overline{\mathrm{R}}$ with $g(\emptyset) = 0$ arise in this way?

*G can be a network with entrance V and exit W.

†$\forall X, Y \subseteq W$, $\forall i \in X \setminus Y$: $g(X) + g(Y) \geq g(X - i) + g(Y + i)$
or $\exists j \in Y \setminus X$: $g(X) + g(Y) \geq g(X - i + j) + g(Y + i - j)$.

Transformation by Graphs/Networks

submodular (set fn)	Y (conjugate formula) matroid $\mapsto$ matroid
separable-conv	N
integrally-conv	N
L-conv ($\mathbb{Z}^n$)	Y (conjugate formula)
M-conv ($\mathbb{Z}^n$)	Y
M-conv (jump)	Y
L-conv (tree^n)	—

Properties

Properties (Structure)

- global min $\Longleftrightarrow$ local min
  - ◇ minimization algorithm
  - ◇ proximity theorem for scaling

- (bi)conjugacy: Legendre transform

  dual object: $f \longrightarrow f^{\bullet} \longrightarrow f^{\bullet\bullet}$
  - ◇ convex extensibility

- separation theorem
  - ◇ convex & concave = linear ?

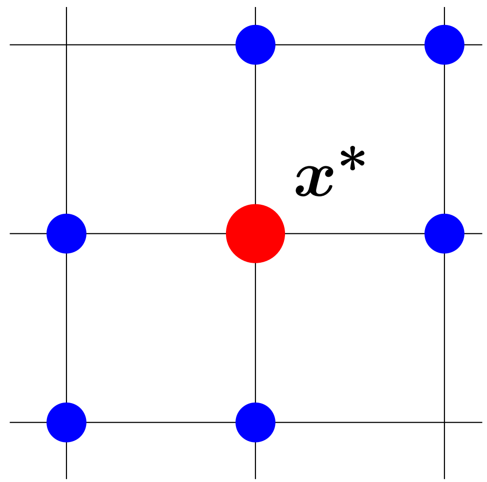
- min-max duality

  min (primal) = max (dual)

Minimization

Local min = Global min

	= ?	#neighbors	poly-time alg	
			local	global
submodular (set fn)	–			
separable-conv	Y			
integrally-conv	Y			
L-conv ($\mathbb{Z}^n$)	Y			
M-conv ($\mathbb{Z}^n$)	Y			
M-conv (jump)	Y			
L-conv (tree^n)	Y			



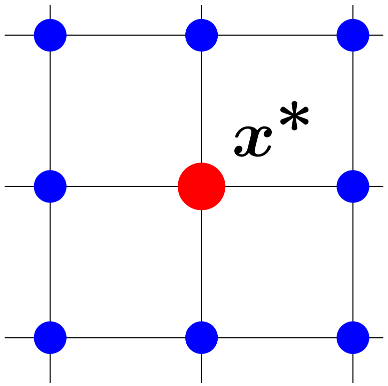
Local vs Global Opt (Integrally-conv)

Thm: (Favati-Tardella 90)

$f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ integrally-convex

x^*: global opt

$\iff$ local opt $f(x^*) \leq f(x^* + y) \quad (\forall y \in \{0, \pm 1\}^n)$



Ex: $x^* + (0, 1, 1, 0, -1, -1, 0, 0)$

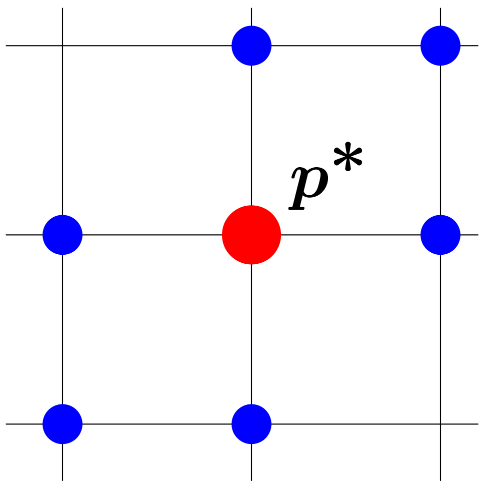
Cannot be checked in poly time

Local vs Global Opt (L♮-conv)

Thm : $g : \mathbb{Z}^n \to \overline{\mathbb{R}}$ L♮-convex (M. 98,03)

p^*: global opt

$\iff$ local opt $g(p^*) \leq g(p^* \pm q)$ $(\forall q \in \{0,1\}^n)$



Ex: $p^* + (0,1,0,1,1,1,0,0)$

$\iff$ $\rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$

takes min at $X = \emptyset$

Can check with n^5 (or n^3) fn evals using submodular fn min algorithm

(Iwata-Fleischer-Fujishige, Schrijver, Orlin)

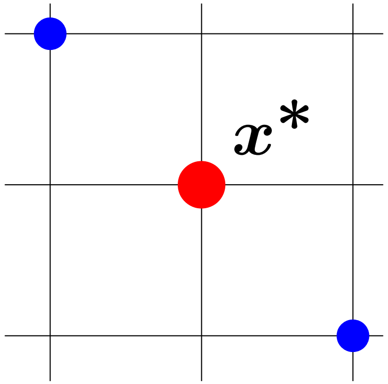
(Lee-Sidford-Wong)

Local vs Global Opt (M-conv)

Thm: $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ M-convex (M. 96)

x^*: global opt

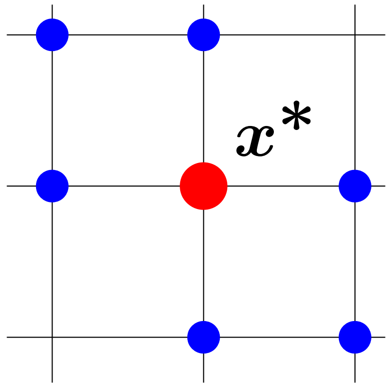
$\iff$ local opt $f(x^*) \leq f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M$^\natural$-convex fn $\Rightarrow$



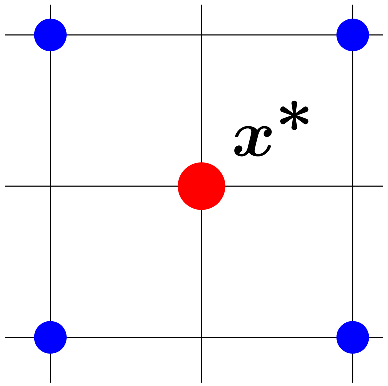
Local vs Global Opt (M-jump)

Thm: (M. 06)

$f : J \to \mathrm{R}$ M-convex on const-parity jump J

x^*: global min

$\iff$ local min $f(x^*) \le f(x^* \pm e_i \pm e_j)$ $(\forall i, j)$



Ex: $x^* + (0, \pm 1, 0, 0, \pm 1, 0, 0, 0)$

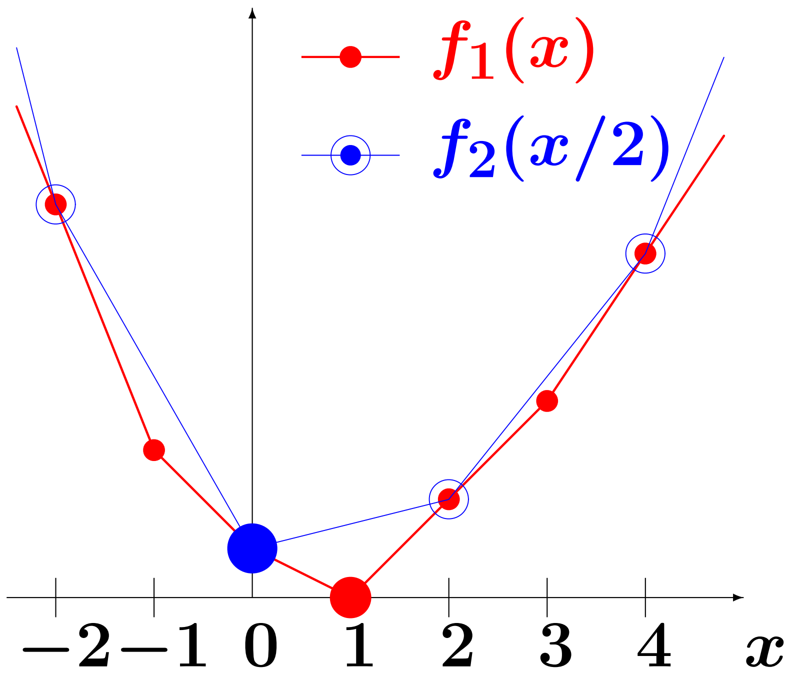
Can check with n^2 fn evals

Local min = Global min

	=	#neigh	poly-time algorithm	
	?	-bors	local opt	global opt
submodular (set fn)	–	—	—	Y
separable-conv	Y	$2n$	Y	Y
integrally-conv	Y	3^n	N	N
L-conv ($\mathbb{Z}^n$)	Y	2^n	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	n^2	Y	Y
M-conv (jump)	Y	$2n^2$	Y	Y
L-conv (tree^n)	Y	graph	Y* (VCSP)	Y* (VCSP)

integrally convex + submodular = L$^\natural$-convex

Scaling and Proximity



Proximity theorem:

True minimum ● exists

in a neighborhood of

a scaled local minimum ●

$\Rightarrow$ efficient algorithm

Facts:

- Scaling preserves L-convexity
- Scaling does NOT preserve M-convexity

Proximity Theorem

	$f(sx)$	proximity theorem
submodular (set fn)	—	—
separable-conv	Y	Y $\|x^* - x^\alpha\|_\infty \leq \alpha - 1$
integrally-conv	Y* ($n = 2$)	Y* ($n = 2$)
L$^\natural$-conv ($\mathbb{Z}^n$)	Y	Y $\|x^* - x^\alpha\|_\infty \leq n(\alpha - 1)$ (Iwata-Shigeno 03)
M$^\natural$-conv ($\mathbb{Z}^n$)	N	Y $\|x^* - x^\alpha\|_\infty \leq n(\alpha - 1)$ (Moriguchi-M.-Shioura 02)
M-conv (jump)	N	?
L-conv (tree^n)	?	Y*

Convex Extension

Convex Extension

submodular (set) ($\mathbb{Z}^n$)	Y N	Lovász extension
separable-conv	Y	$\overline{f}(x) = \overline{\varphi_1}(x_1) + \cdots + \overline{\varphi_n}(x_n)$
integrally-conv	Y	by definition
L-conv ($\mathbb{Z}^n$)	Y	Lovász extension, locally
M-conv ($\mathbb{Z}^n$)	Y	non-constructive
M-conv (jump)	N	
L-conv (tree^n)	Y	in an appropriate sense (Euclidean building, CAT(0))

Conjugacy

Conjugacy/Duality in Matroids

Conjugacy

Exchange axiom $\Leftrightarrow$ Submodularity of rank function

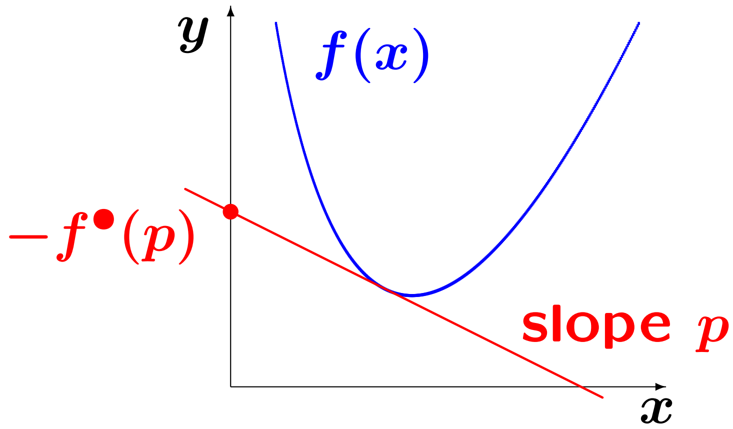
Duality

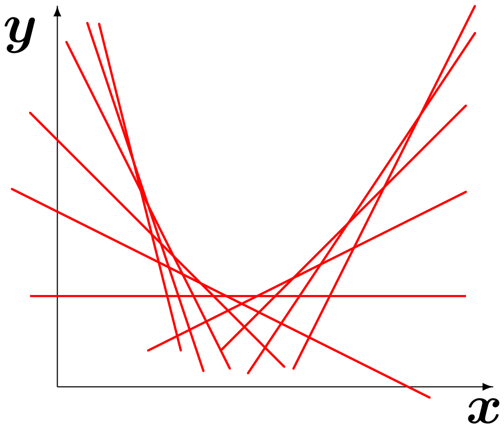
Matroid intersection theorem (Edmonds)

Discrete separation (Frank)

Fenchel-type duality (Fujishige)

Conjugacy: Discrete Legendre Transform





$$f^{\bullet}(p) = \sup_{x \in \mathbf{Z}^n} \{\langle p, x \rangle - f(x)\}$$

$\Rightarrow$ **If** $f : \mathrm{Z}^n \to \overline{\mathrm{Z}}$, **then** $f^{\bullet} : \mathrm{Z}^n \to \overline{\mathrm{Z}}$

(integer-valued)

Conjugacy in Polymatroids

Polyhedron S		Submodular fn ρ
$S = \{x \mid x(A) \le \rho(A)\ \ \forall A\}$	$\leftarrow$	
	$\rightarrow$	$\rho(A) = \max_{x \in S} x(A)$

Conjugacy in Polymatroids

Polyhedron S		**Submodular fn** ρ
$S = \{x \mid x(A) \le \rho(A) \ \forall A\}$	$\leftarrow$	
	$\rightarrow$	$\rho(A) = \max_{x \in S} x(A)$

Indicator fn of S — **Lovász ext. of** ρ

$f(x) \in \{0, +\infty\}$ — $g(p)$

$\rightarrow$: $g(p) = \max_{x \in S} \langle p, x \rangle = \max_{x} [\langle p, x \rangle - f(x)] = f^{\bullet}(p)$

$\leftarrow$: $f(x) = \max_{p} [\langle p, x \rangle - g(p)] = g^{\bullet}(x)$

Legendre transform

M$^\natural$-convex — L$^\natural$-convex

M-L Conjugacy Theorem

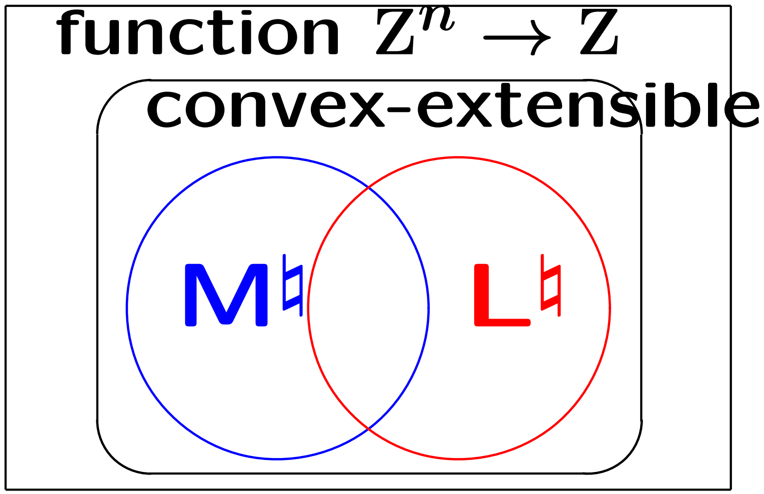
Integer-valued discrete fn $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

Legendre transform: $f^\bullet(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

(1) M and L are conjugate **(M. 98)**

(2) M♮ and L♮ are conjugate

$$f \mapsto f^\bullet = g \mapsto g^\bullet = f$$



(3) biconjugacy

$$f^{\bullet\bullet} = f$$

for $f \in$ **M♮** $\cup$ **L♮**

History of Discrete Conjugacy

Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	**Submodular fn**
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid	|	Lovász extension
Dress–Wenzel 90	|	Lovász 83
$\Downarrow$	|	$\Downarrow$
	|	Submod. integ. conv. fn
	|	Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	**L-convex fn**
M. 96		M. 98
$\Updownarrow$		$\Updownarrow$
$\mathrm{M}^\natural$-convex fn	$\longleftrightarrow$	$\mathrm{L}^\natural$-convex fn
M. -Shioura 99		Fujishige–M. 00

Conjugacy and Biconjugacy

$f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$	$f^\bullet : \mathbf{Z}^n \to \overline{\mathbf{Z}}$	$f^{\bullet\bullet} = f$
submodular (set fn)	submodular polyhedron $\{x \in \mathbf{Z}^n \mid x(A) \leq \rho(A)\}$	Y
separable-conv $f(x) = \sum \varphi_i(x_i)$	$\varphi_1^\bullet(p_1) + \cdots + \varphi_n^\bullet(p_n)$	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	M-convex	Y
L$^\natural$-conv	M$^\natural$-convex	Y
M-conv ($\mathbb{Z}^n$)	L-convex	Y
M$^\natural$-conv	L$^\natural$-convex	Y
M-conv (jump)	N	N
L-conv (tree^n)	—	—

Significance of Conjugacy

- Electrical network $\leftarrow$ **Iri's book 69**

  x: current, p: voltage (potential)

- Economics (auction)

  x: commodity bundle, p: price vector

- Discrete DC programming (Maehara-M. 15)

  Toland–Singer duality

Duality

Convex & Concave = Linear

f, $-f$: convex $\Longrightarrow f(x) = \langle p, x \rangle + \alpha$

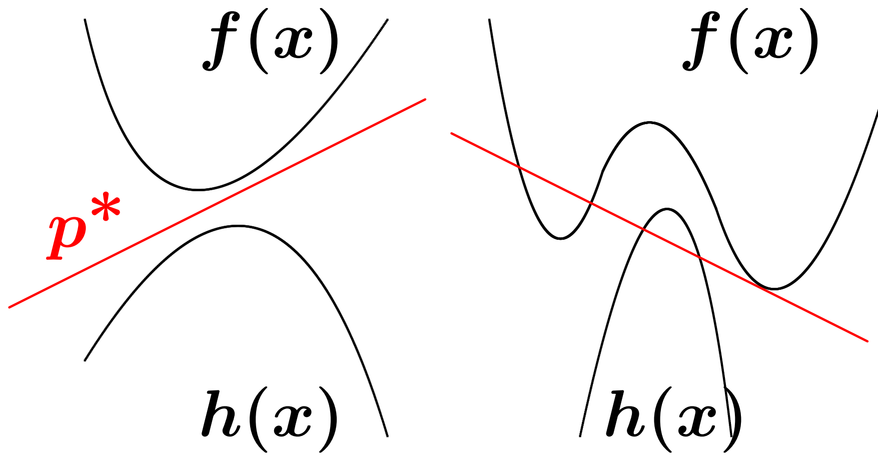
submodular (set)	Y
($\mathbb{Z}^n$)	N
separable-conv	Y
integrally-conv	N (every set fn is int-conv)
L-conv ($\mathbb{Z}^n$)	Y
M-conv ($\mathbb{Z}^n$)	Y
M-conv (jump)	N (example by Yabe)
L-conv (tree^n)	Y* (constant)

N $\Longrightarrow$ no separation theorem

Discrete Separation Theorem

$f : \mathbf{Z}^n \to \mathbf{R}$ **"convex"**

$h : \mathbf{Z}^n \to \mathbf{R}$ **"concave"**



- $f(x) \geq h(x) \quad (\forall x \in \mathbf{Z}^n) \Rightarrow \ \exists\ \alpha^* \in \mathbf{R}, \ \ \exists\ p^* \in \mathbf{R}^n$:

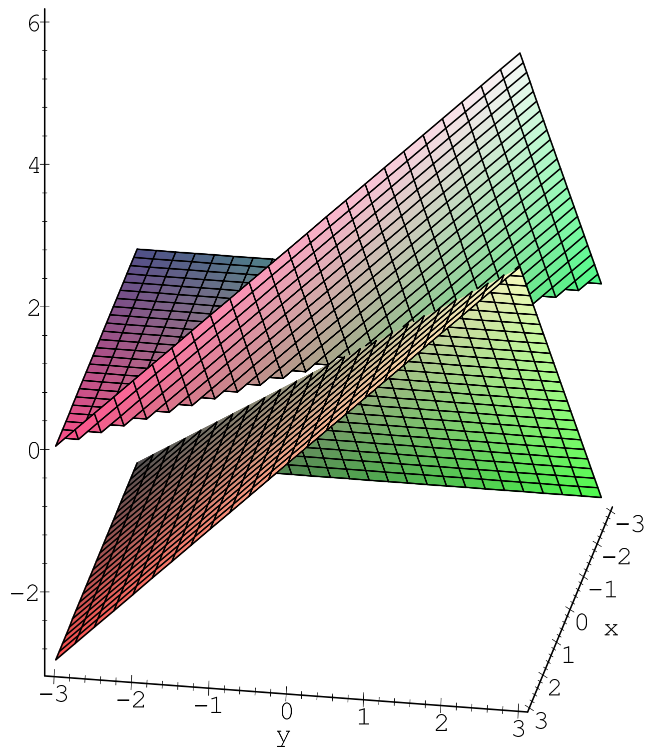
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbf{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbf{Z}, \quad p^* \in \mathbf{Z}^n$

Difficulty of Discrete Separation (1)

$f(x,y) = \max(0, x+y)$ **convex**

$h(x,y) = \min(x,y)$ **concave**



nonintegral separation

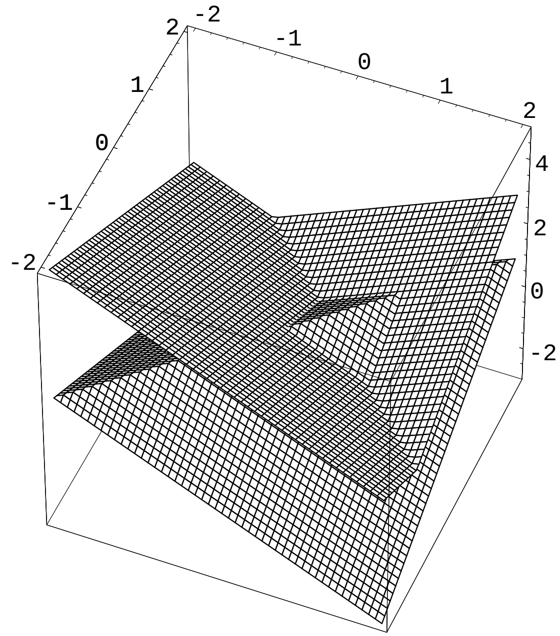
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x,y) = |x+y-1|$ **convex**

$h(x,y) = 1-|x-y|$ **concave**



- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathrm{Z}^2)$ **true**
- **No** $\alpha^* \in \mathrm{R}$, $p^* \in \mathrm{R}^2$: $f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x)$

$\because$ $f = 0 < h = 1$ **at** $(x,y) = (1/2, 1/2)$

Frank's Discrete Separation

(Frank 82)

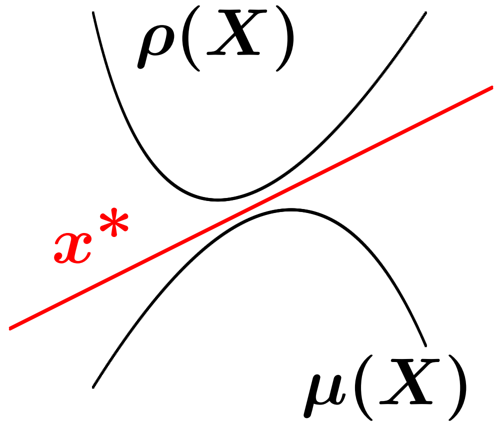
$\rho : 2^V \to \mathbf{R}$: **submodular** $\qquad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathbf{R}$: **supermodular** $\qquad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathbf{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- ρ, μ: **integer-valued** $\Rightarrow x^* \in \mathbf{Z}^V$



Discrete Separation Theorems

(M. 96/98)

M-separation Thm (for M$^\natural$-convex)

$\Rightarrow$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn = M$^\natural$-convex fn)

L-separation Thm (for L$^\natural$-convex)

$\Rightarrow$ Discrete separ. for submod. set fn (Frank 82)

(submod. set fn = L$^\natural$-convex fn on 0–1 vectors)

Separation and Min-Max Theorems

	separation	min-max
submodular (set fn)	Y (Frank)	Y (Edmonds, Fujishige)
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	Y
M-conv (jump)	N	N
L-conv (tree^n)	?	?

Edmonds' Intersection Theorem

Submodular polyhedron $(\rho(\emptyset) = 0,\ \rho(V) < +\infty)$

$$\mathrm{P}(\rho) = \{x \in \mathrm{R}^n \mid x(X) \leq \rho(X)\ (\forall X \subseteq V)\} \qquad (|V| = n)$$

Thm: **(Edmonds 70)**

(1) For $\rho_1, \rho_2 : 2^V \to \overline{\mathrm{R}}$: **submodular,**

$$\max_x \{x(V) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2)\} = \min_X \{\rho_1(X) + \rho_2(V \setminus X)\}.$$

(2) If ρ_1 **and** ρ_2 **are integer-valued, then**

$$\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) = \overline{\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) \cap \mathrm{Z}^n}$$

and there exists $x^* \in \mathrm{Z}^n$ **that attains the maximum.**

Min-Max Duality

f: M$^\natural$-convex, h: M$^\natural$-concave $(\mathbf{Z}^n \to \overline{\mathbf{Z}})$

Legendre–Fenchel transform

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\}$$

Fenchel-type duality thm (M. 96, 98)

$$\inf_{x \in \mathbf{Z}^n}\{f(x) - h(x)\} = \sup_{p \in \mathbf{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

self-conjugate ($f^\bullet$: L$^\natural$-convex, h°: L$^\natural$-concave)

$\Longrightarrow$ Edmonds' intersection thm
Fujishige's Fenchel dualtiy thm

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation	
$f(x) \geq \boxed{\text{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84) matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\}$ $= \sup\{h^\circ - f^\bullet\}$	$\Rightarrow$ discrete separ. for submod (Frank 82) $\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation	weighted matroid intersect.
$f^\bullet(p) \geq \boxed{\text{Lin}} \geq h^\circ(p)$	(Edmonds 79, Iri-Tomizawa 76, Frank 81)

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	—	Y	N	Y*	—	—	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	—	—	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca ling	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi-cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	–	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi-cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	–	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	–	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

END

Supplement

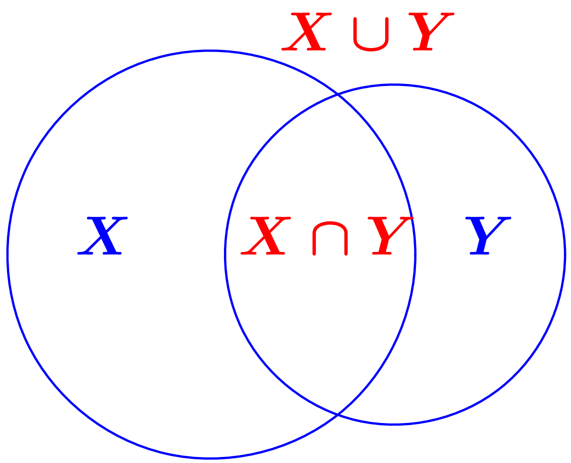
Submodular Function

$$\overline{\mathrm{R}} = \mathrm{R} \cup \{+\infty\}$$

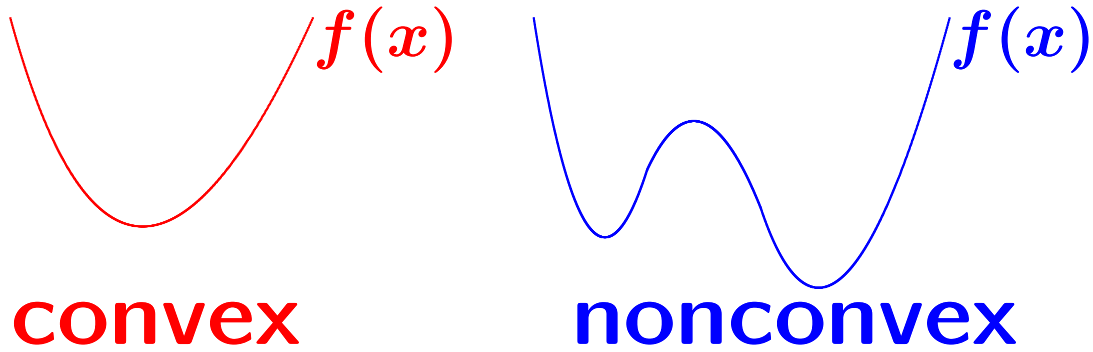
Set function $\rho : 2^V \to \overline{\mathrm{R}}$ **is submodular**

$\Longleftrightarrow$

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

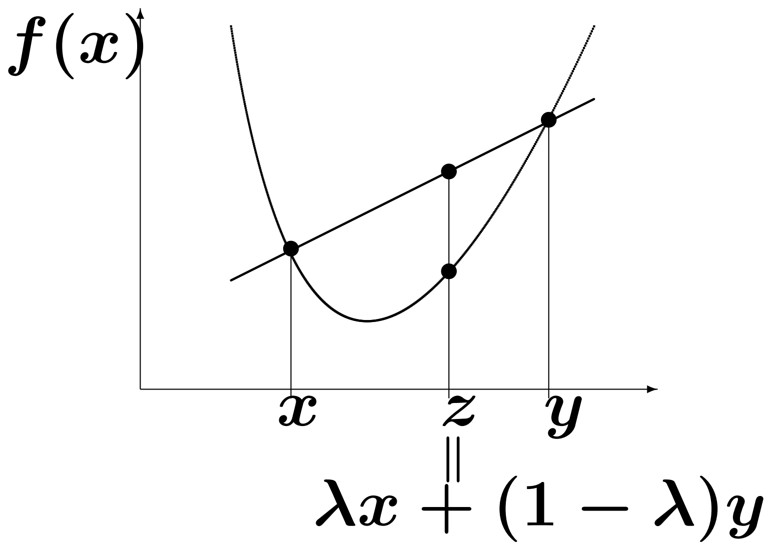


Convex Function



$f : \mathrm{R}^n \to \overline{\mathrm{R}}$ is convex $\iff$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \qquad (0 < \forall\ \lambda < 1)$$



Scaling/Linear Addition

	$af(x)$ $(a > 0)$	$f(sx)$ $(s \in \mathbb{Z}_+)$	$f(-x)$	$f(x)+$ $\langle p, x\rangle$
submodular (set fn)	Y	—	Y* $V \setminus X$	Y
separable-conv	Y	Y	Y $\pm x_i$	Y
integrally-conv	Y	Y* $(n = 2)$	Y $\pm x_i$	Y
L-conv $(\mathbb{Z}^n)$	Y	Y	Y	Y
M-conv $(\mathbb{Z}^n)$	Y	N	Y	Y
M-conv (jump)	Y	N	Y $\pm x_i$	Y
L-conv (star^n)	Y	?	—	Y* dist(0,·)

Section/Projection

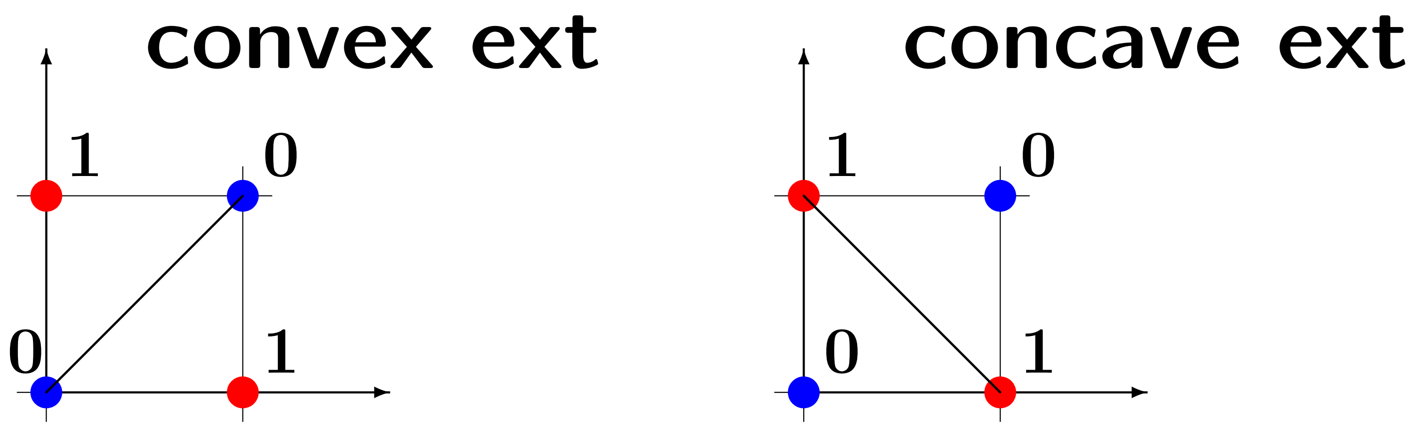
	section $f(x,0)$	projection $\min_y f(x,y)$
submodular (set fn)	Y restriction	Y contraction
separable-conv	Y	Y
integrally-conv	Y	?
L-conv ($\mathbb{Z}^n$)	N	Y
L$^\natural$-conv	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	N
M$^\natural$-conv	Y	Y
M-conv (jump)	Y	Y
L-conv (tree^n)	Y	Y

Set Function and Extensions

Set function $\Longleftrightarrow$ Function on $\{0,1\}^n$

$$\rho(X) \quad = \quad \hat{\rho}(\chi_X)$$

Every set function $\rho : \{0,1\}^n \to \mathrm{R}$ can be extended to convex/concave function



cf. Lovász extension

Convex Extension

submodular (set) ($\mathbb{Z}^n$)	Y N	Lovász extension
separable-conv	Y	$\overline{f}(x) = \overline{\varphi_1}(x_1) + \cdots + \overline{\varphi_n}(x_n)$
integrally-conv	Y	by definition
L-conv ($\mathbb{Z}^n$)	Y	Lovász extension, locally
M-conv ($\mathbb{Z}^n$)	Y	non-constructive
M-conv (jump)	N	
L-conv (tree^n)	Y	in an appropriate sense (Euclidean building, CAT(0))

Dual Character of Matroid Rank Func

$\rho(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$

is **L$^\natural$-convex** and **M$^\natural$-concave**

Self-Conjugacy: $\rho(X) = |X| - \rho^\bullet(\chi_X)$

Prf: $\rho^\bullet(\chi_X) = \max_Y \{|X \cap Y| - \rho(Y)\}$

$= \max_{Y \supseteq X} \{|X \cap Y| - \rho(Y)\} = |X| - \rho(X)$

ρ subm $\Rightarrow$ ρ L$^\natural$-conv $\Rightarrow$ $\rho^\bullet$ M$^\natural$-conv $\Rightarrow$ ρ M$^\natural$-concave

Edmonds's matroid union formula:

$$\max_X \{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y \{\rho_1(Y) + \rho_2(Y) + |V \setminus Y|\}$$

submod maximization (M$^\natural$-concave + M$^\natural$-concave)

submod minimization (L$^\natural$-convex + L$^\natural$-convex)

Polymatroid Rank Function

Polymatroid rank function is NOT M$^\natural$-concave

Example $\rho : 2^V \to \mathbf{Z}$ on $V = \{1, 2, 3, 4\}$ (Shioura)

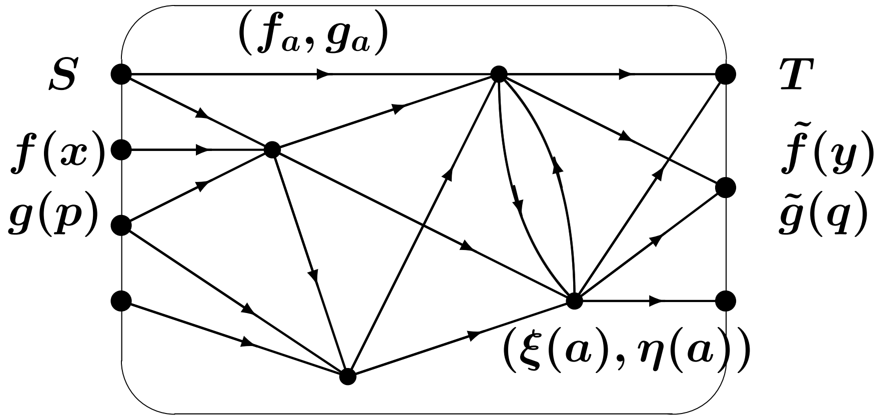
$\rho(\emptyset) = 0, \quad \rho(i) = 2 \ (i \in V), \quad \rho(1, 2) = \rho(3, 4) = 4,$

$\rho(1, 3) = \rho(1, 4) = \rho(2, 3) = \rho(2, 4) = 3,$

$\rho(X) = 4$ if $|X| \geq 3$

Exchange fails for $X = \{1, 2\}$, $Y = \{3, 4\}$

Trans. by Network



M-convex ($\mathbb{Z}^n$): $\qquad (y \in \mathbf{Z}^T)$

$$\tilde{f}(y) = \inf_{\xi, x} \{ f(x) + \sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = (x, -y, 0), \quad \xi \in \mathbf{Z}^A, (x, -y, 0) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V \setminus (S \cup T)} \}$$

L-convex ($\mathbb{Z}^n$): $\qquad (q \in \mathbf{Z}^T)$

$$\tilde{g}(q) = \inf_{\eta, p, r} \{ g(p) + \sum_{a \in A} g_a(\eta(a)) \mid \eta = -\delta(p, q, r), \quad \eta \in \mathbf{Z}^A, (p, q, r) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V \setminus (S \cup T)} \}$$

M-convex & M-concave (jump) $\neq$ Linear

(by A. Yabe)

$V = \{1,2,3,4\}$, $\mathcal{F} = \{X \mid |X| = \text{even}\}$, $f : \mathcal{F} \to \mathbb{R}$:

$$f(\emptyset) = 0, \quad f(\{1,2\}) = f(\{3,4\}) = 2, \quad f(\{1,2,3,4\}) = 4,$$

$$f(\{1,3\}) = f(\{1,4\}) = f(\{2,3\}) = f(\{2,4\}) = a \ (\neq 2)$$

- $(V, \mathcal{F})$ is even delta-matroid
- For $\forall X, Y \in \mathcal{F}$, $\forall i \in X \Delta Y$, $\exists j \in (X \Delta Y) \setminus \{i\}$:

$$f(X) + f(Y) = f(X \Delta \{i,j\}) + f(Y \Delta \{i,j\})$$

- f, $-f$: M-convex (jump)
- NO (p, α): $f(x) = \langle p, x \rangle + \alpha$ (i.e., not linear)

Edmonds' Intersection Theorem

Submodular polyhedron $(\rho(\emptyset) = 0,\ \rho(V) < +\infty)$

$$\mathrm{P}(\rho) = \{x \in \mathrm{R}^n \mid x(X) \leq \rho(X)\ (\forall X \subseteq V)\} \qquad (|V| = n)$$

Thm: (Edmonds 70)

(1) For $\rho_1, \rho_2 : 2^V \to \overline{\mathrm{R}}$**: submodular,**

$$\max_x \{x(V) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2)\} = \min_X \{\rho_1(X) + \rho_2(V \setminus X)\}.$$

(2) If ρ_1 **and** ρ_2 **are integer-valued, then**

$$\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) = \overline{\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) \cap \mathrm{Z}^n}$$

and there exists $x^* \in \mathrm{Z}^n$ **that attains the maximum.**

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation $f(x) \geq \boxed{\text{Lin}} \geq h(x)$	**Fenchel duality** (Fujishige 84) **matroid intersect.** (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\} = \sup\{h^{\circ} - f^{\bullet}\}$	$\Rightarrow$ **discrete separ. for submod** (Frank 82) $\Rightarrow$ **valuated matroid intersect.** (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation $f^{\bullet}(p) \geq \boxed{\text{Lin}} \geq h^{\circ}(p)$	**weighted matroid intersect.** (Edmonds 79, Iri-Tomizawa 76, Frank 81)

E N D