Hausdorff School: Economics and Tropical Geometry
Bonn, May 9-13, 2016

Discrete Convex Analysis I:

Concepts of Discrete Convex Functions

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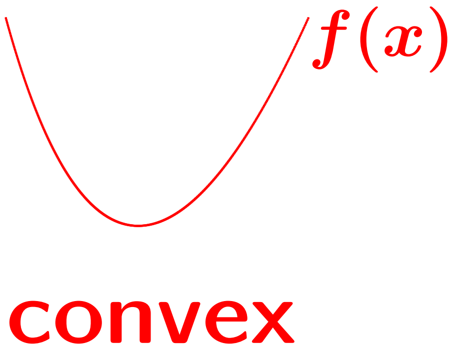
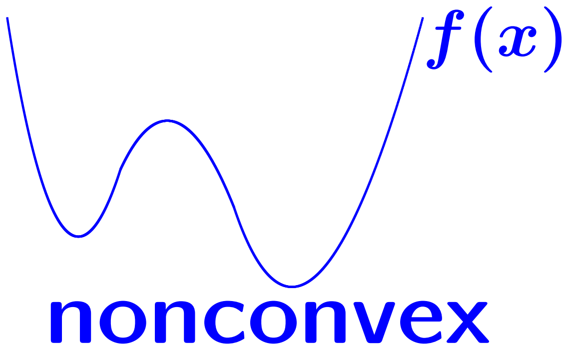
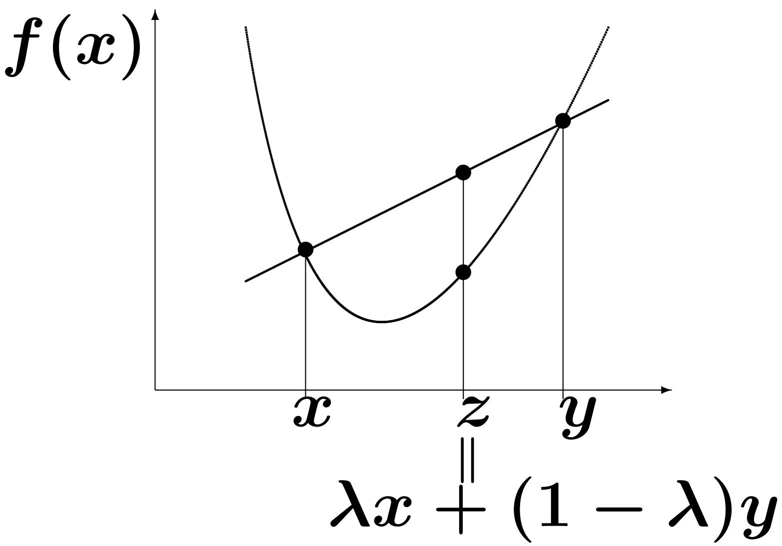
Convex Function





$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ **is convex** $\iff$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \qquad (0 < \forall\ \lambda < 1)$$



$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

Features of Convex Functions

- Occurrence in many models

  motivations, applications

- Operations and transformations

- Sufficient structure for a theory

  mathematically beautiful, practically useful

- Minimization algorithms

Features of Convex Functions

= Issues in discrete convex analysis

- Occurrence in many models ?

  motivations, applications

- Operations and transformations ?

- Sufficient structure for a theory ?

  mathematically beautiful, practically useful

- Minimization algorithms ?

Contents of Part I

Concepts of Discrete Convex Functions

C1. Univariate Discrete Convex Functions

C2. Classes of Discrete Convex Functions

C3. L-convex Functions

C4. M-convex Functions

C5. Remarks on Submodular Set Functions

Part II: Properties, Part III: Algorithms

C1.
Univariate Discrete Convex Functions

Ingredients of convex analysis

Definition of Convex Function

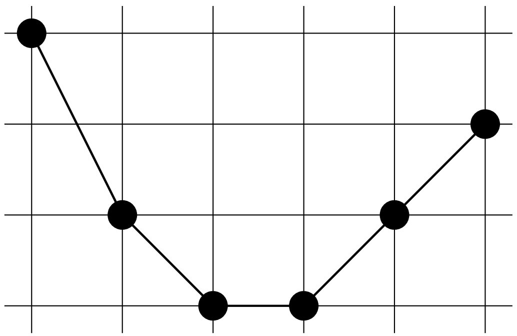
$f : \mathbb{Z} \to \overline{\mathbb{R}}$ $\qquad\qquad \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

$$f(x-1) + f(x+1) \geq 2f(x)$$

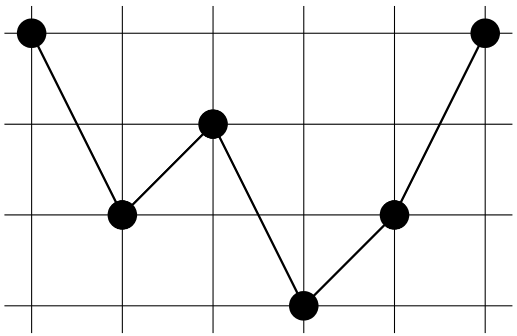
$$\iff \quad f(x) + f(y) \geq f(x+1) + f(y-1) \qquad (x < y)$$

$\iff$ f **is convex-extensible, i.e.,**

$\exists$ **convex** $\overline{f} : \mathbb{R} \to \overline{\mathbb{R}}$ **s.t.** $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z})$



convex



non-convex

Local vs Global Optimality

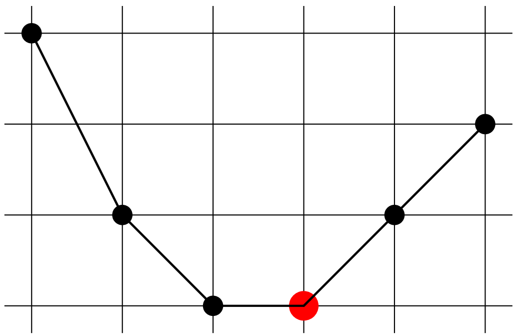
$f : \mathbb{Z} \to \overline{\mathbb{R}}$

Theorem:

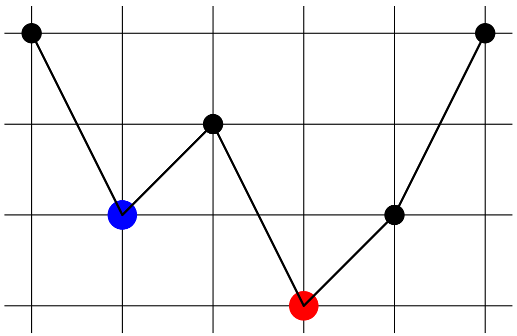
x^*: global opt (min)

$\iff$ x^*: local opt (min)

$$f(x^*) \le \min\{f(x^* - 1),\ f(x^* + 1)\}$$

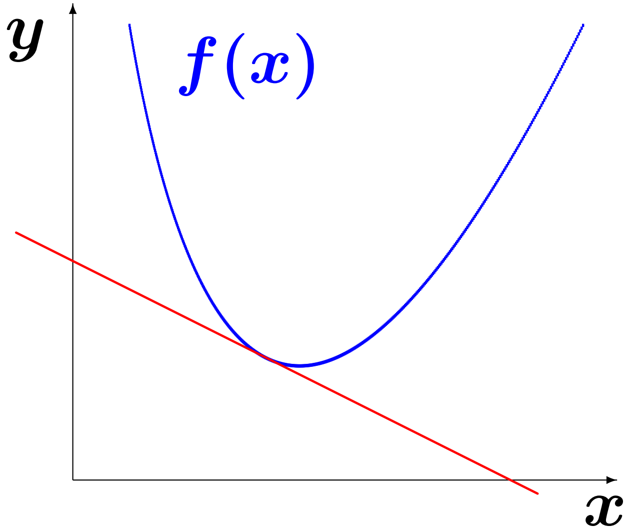


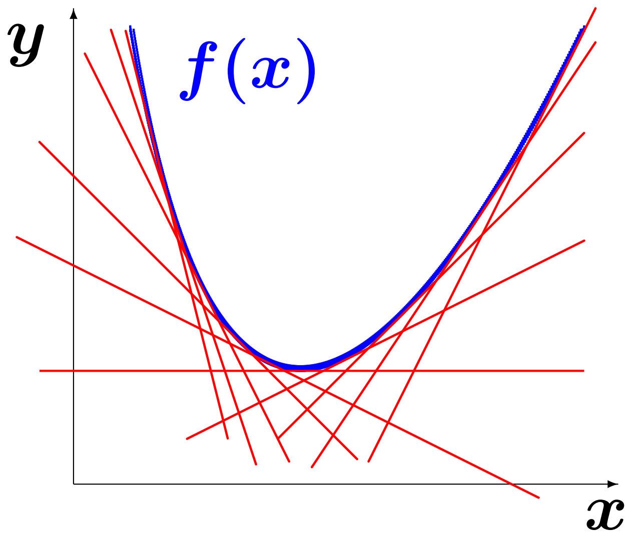
convex



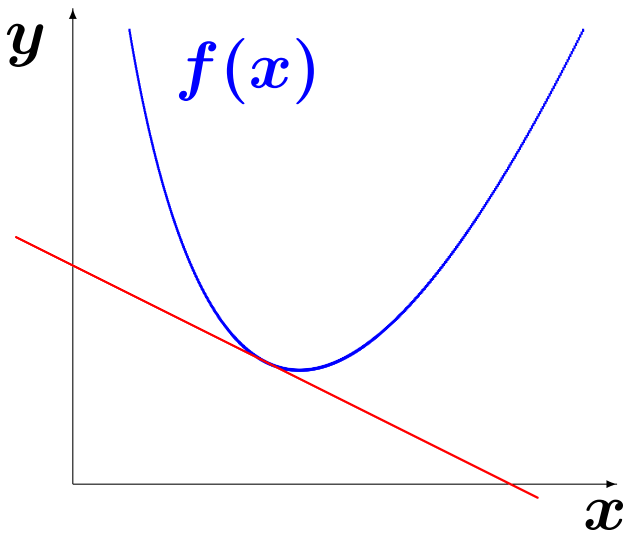
non-convex

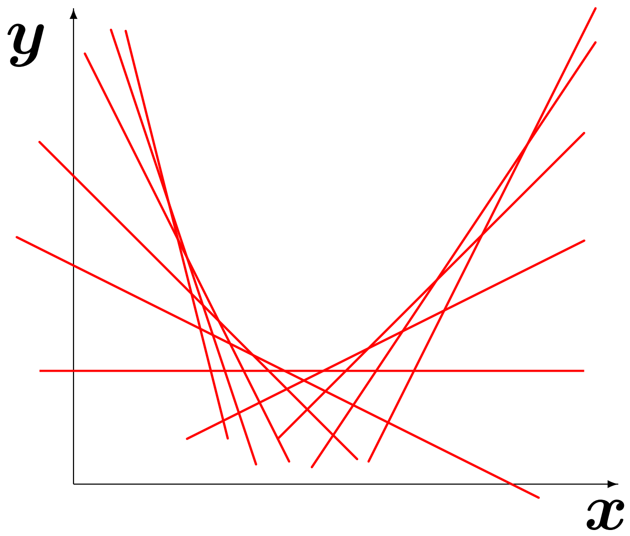
Intuition of Legendre Transformation





Intuition of Legendre Transformation





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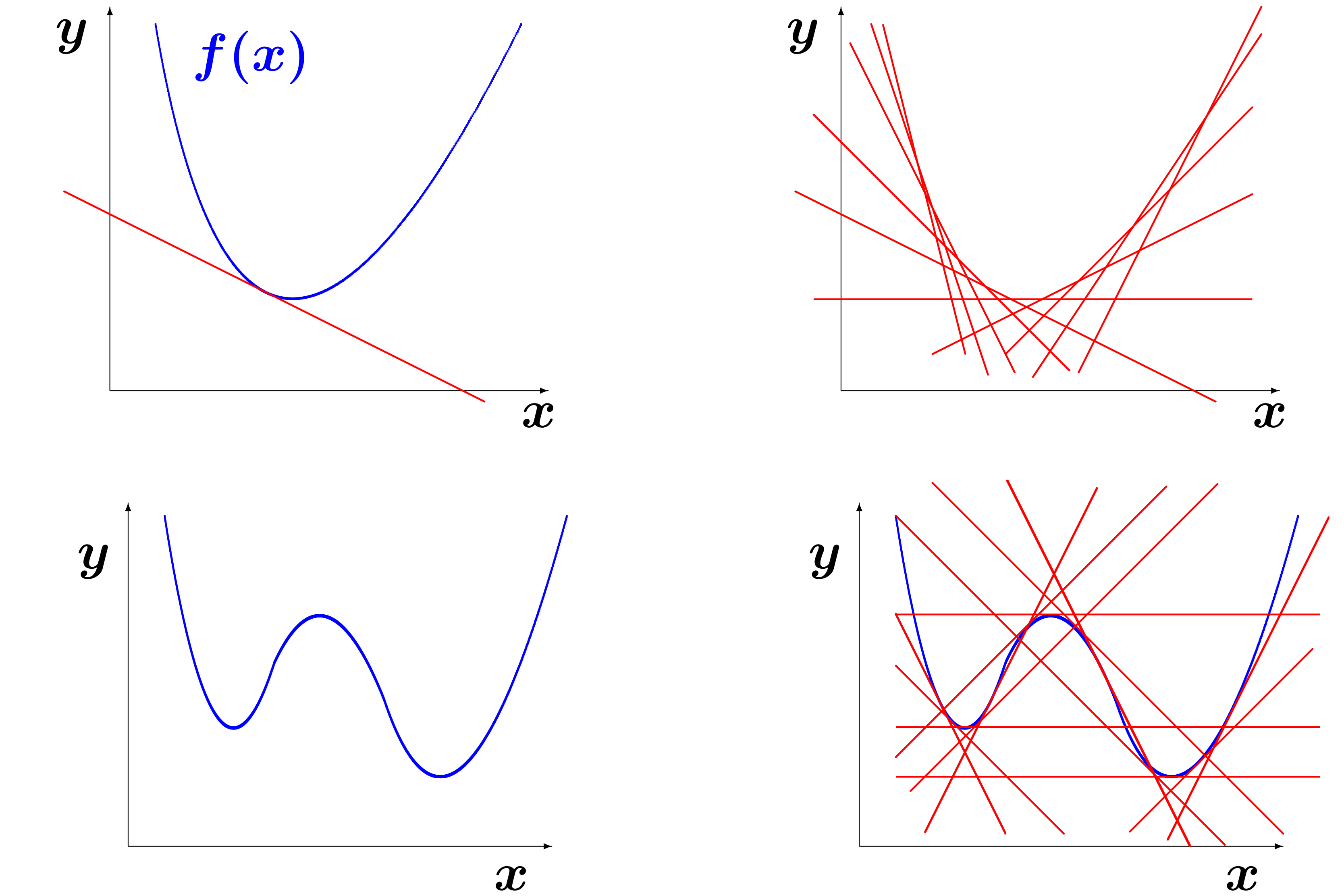
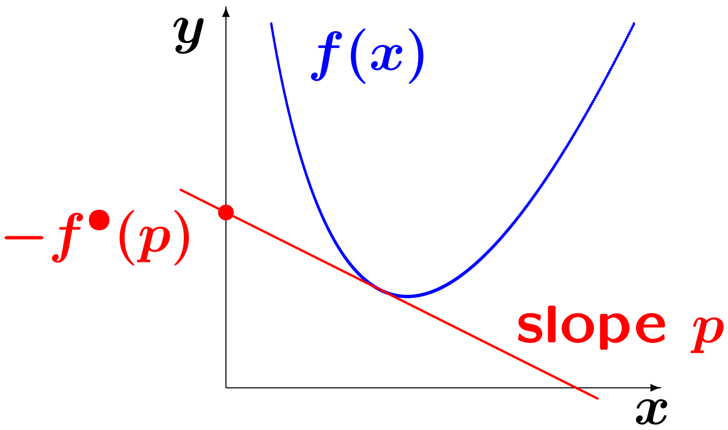


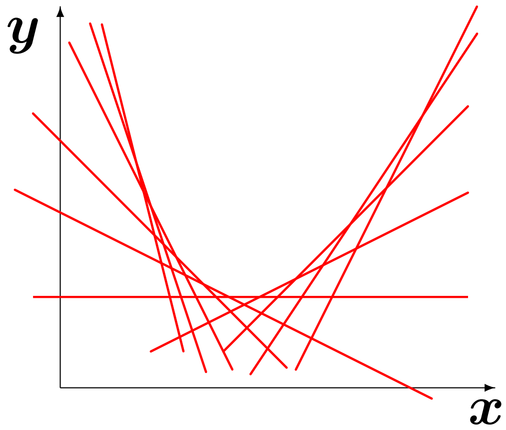
Legendre Transformation

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$ (integer-valued)

Define discrete Legendre transform of f by

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z})$$





Theorem:

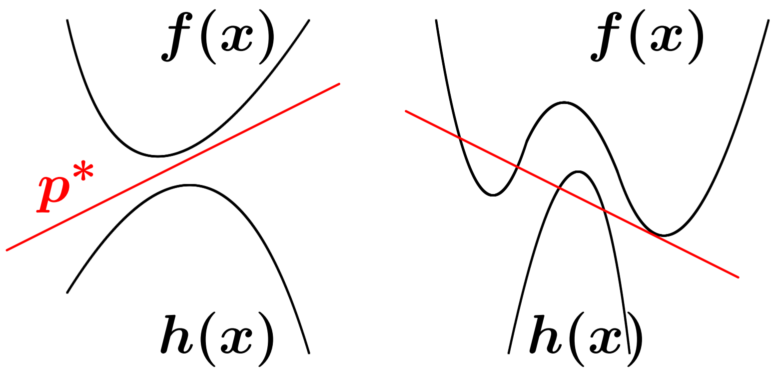
(1) $f^{\bullet}$ is $\mathbb{Z}$-valued convex function, $f^{\bullet} : \mathbb{Z} \to \overline{\mathbb{Z}}$

(2) $(f^{\bullet})^{\bullet} = f$ (biconjugacy)

Separation Theorem

$f : \mathbb{Z} \to \overline{\mathbb{R}}$
convex

$h : \mathbb{Z} \to \underline{\mathbb{R}}$
concave



Theorem (Discrete Separation Theorem)

(1) $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}) \Rightarrow \ \exists\, \alpha^* \in \mathbb{R}, \ \exists\, p^* \in \mathbb{R}$:

$$f(x) \geq \alpha^* + p^* x \geq h(x) \quad (\forall x \in \mathbb{Z})$$

(2) f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \ \ p^* \in \mathbb{Z}$

Separation Theorem

$f : \mathbb{Z} \to \overline{\mathbb{R}}$
convex

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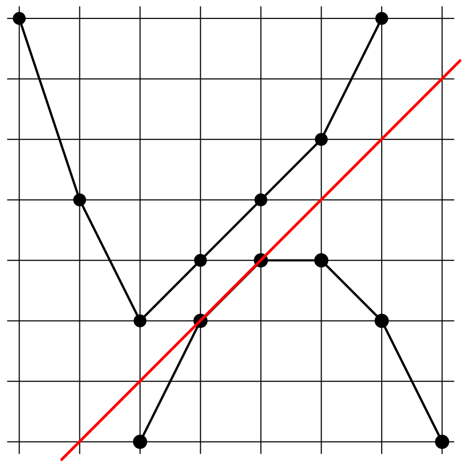

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Fenchel Duality (Min-Max)

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$: convex, $\quad h : \mathbb{Z} \to \underline{\mathbb{Z}}$: concave

Legendre transforms:

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\}$$

$$h^{\circ}(p) = \inf\{px - h(x) \mid x \in \mathbb{Z}\}$$

Theorem:

$$\inf_{x \in \mathbb{Z}} \{f(x) - h(x)\} = \sup_{p \in \mathbb{Z}} \{h^{\circ}(p) - f^{\bullet}(p)\}$$

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

hold for univariate discrete convex functions

C2.

Classes of

Discrete Convex Functions

Classes of Discrete Convex Functions

1. Submodular set fn (on $\{0, 1\}^n$)
1. Separable-convex fn on $\mathbb{Z}^n$
1. Integrally-convex fn on $\mathbb{Z}^n$

2. L-convex (L$^\natural$-convex) fn on $\mathbb{Z}^n$
2. M-convex (M$^\natural$-convex) fn on $\mathbb{Z}^n$

3. M-convex fn on jump systems
3. L-convex fn on graphs

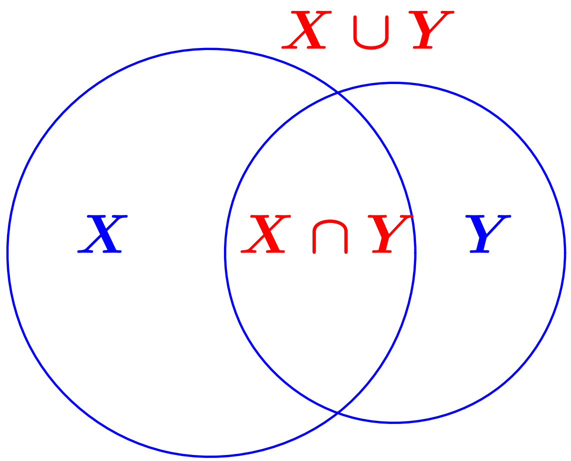
Submodular Function

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Set function $\rho : 2^V \to \overline{\mathbb{R}}$ **is submodular**

$\iff$

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



cf. $|X| + |Y| = |X \cup Y| + |X \cap Y|$

Set function $\iff$ **Function on** $\{0,1\}^n$

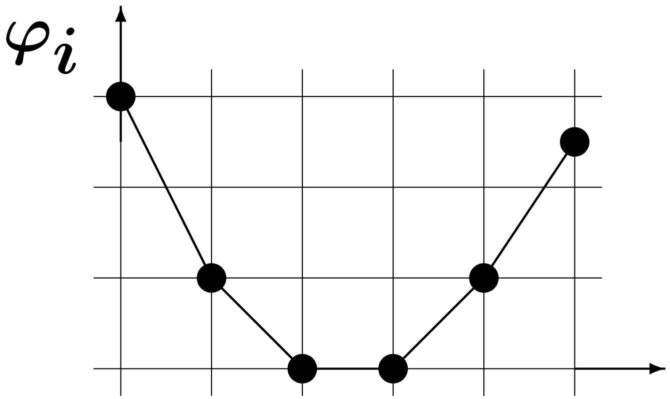
Separable-convex Function

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is separable-convex

$\iff$

$$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$$

φ_i: univariate convex



Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

hold for separable discrete convex functions

Some History

1935	Matroid	Whitney, Nakasawa
1965	Submodular function	Edmonds
1969	Convex network flow (electr.circuit)	Iri
1982	Submodularity and convexity	Frank, Fujishige, Lovász
1990	Valuated matroid	Dress–Wenzel
	Integrally convex fn	Favati–Tardella
1996	Discrete convex analysis	Murota
2000	Submodular minimization algorithm	Iwata–Fleischer–Fujishige, Schrijver
2006	M-convex fn on jump system	Murota
2012	L-convex fn on graph	Hirai, Kolmogorov

Motivations/Applications/Connections

1. submodular	MANY problems graph cut, convex game
1. separable-conv	MANY problems min-cost flow, resource allocation
1. integrally-conv	[mathematical aesthetics]
2. L-conv ($\mathbb{Z}^n$)	network tension, image processing OR (inventory, scheduling)
2. M-conv ($\mathbb{Z}^n$)	network flow, conjestion game economics (game, auction) mixed polynomial matrix
3. M-conv (jump)	deg sequence, (2-)matching polynomial (half-plane property)
3. L-conv (graph)	multiflow, multifacility location

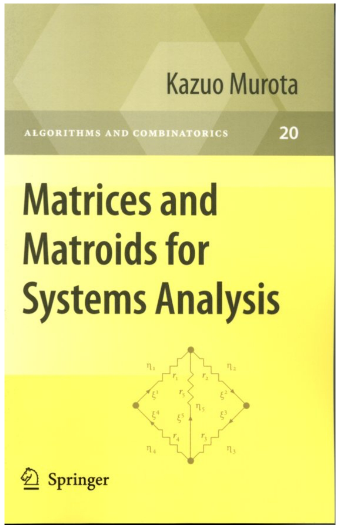
Books (discrete convex analysis)

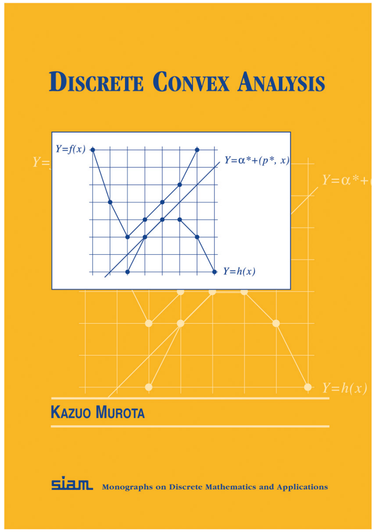
2000: Murota, Matrices and Matroids for Systems Analysis, Springer

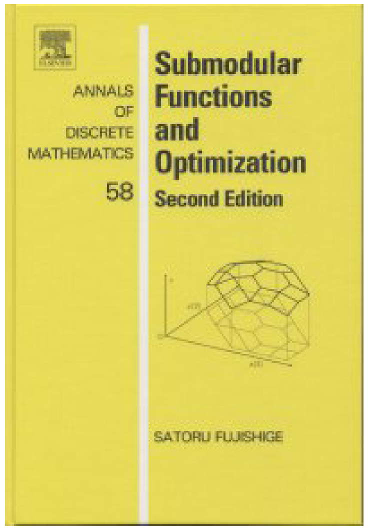
2003: Murota, Discrete Convex Analysis, SIAM

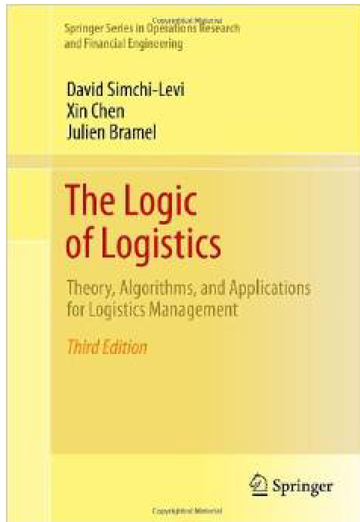
2005: Fujishige, Submodular Functions and Optimization, 2nd ed., Elsevier

2014: Simchi-Levi, Chen, Bramel, The Logic of Logistics, 3rd ed., Springer









Convex Extension

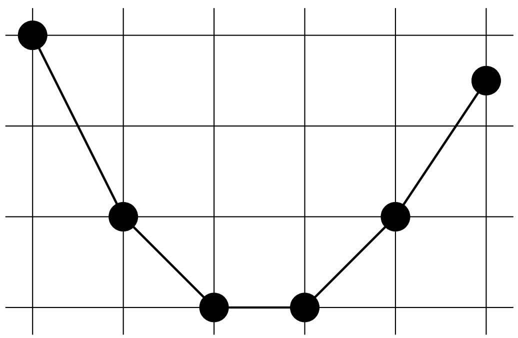
- $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex-extensible

  $\Leftrightarrow$ $\exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

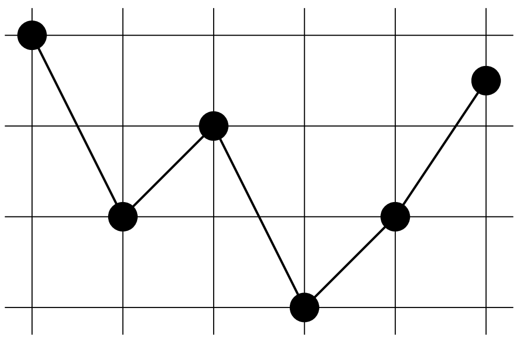
- $\overline{f}$ is a convex extension of f

- convex closure

  = (pointwise) max convex extension



convex-extensible

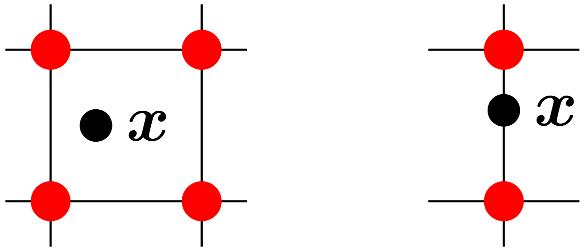


NOT convex-extensible

Integrally Convex Function

(Favati-Tardella 1990)

$$N(x) = \{y \in \mathbb{Z}^n \mid \|x - y\|_\infty < 1\} \qquad (x \in \mathbb{R}^n)$$

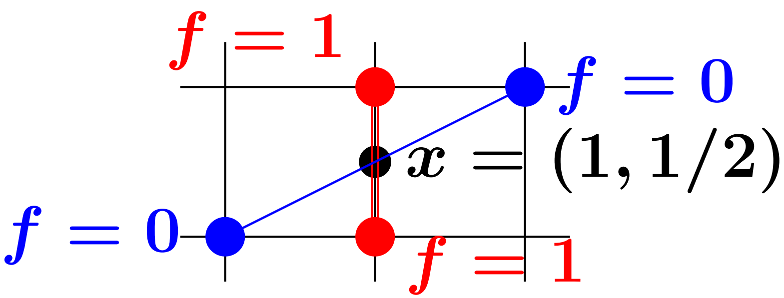


Local convex extension:

$$\tilde{f}(x) = \sup_{p,\alpha}\{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \le f(y) \ (\forall y \in N(x))\}$$

Def: f is integrally convex $\iff$ $\tilde{f}$ is convex

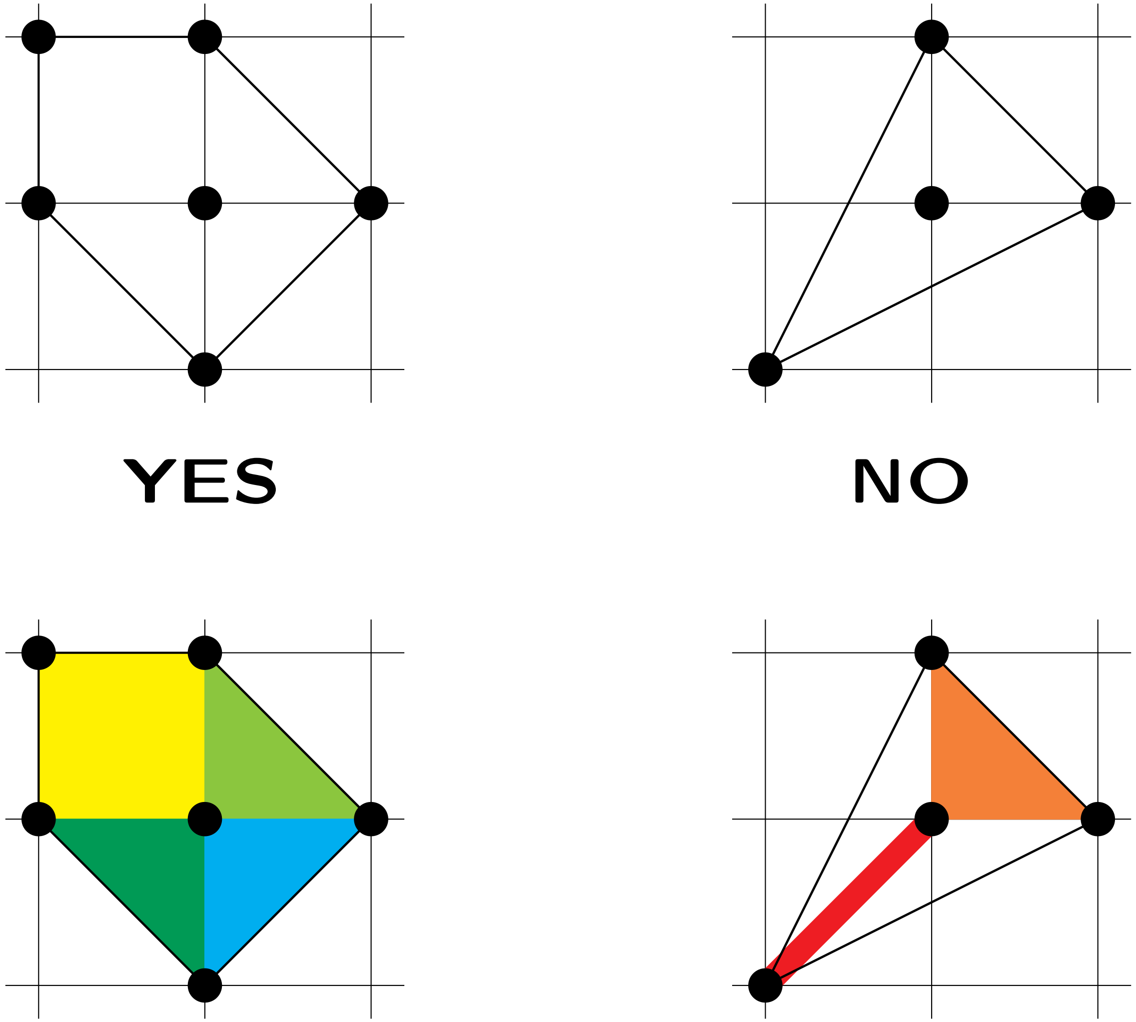
Ex: $f(x_1, x_2) = |x_1 - 2x_2|$ is NOT integrally convex



$\tilde{f}(x) = 1$

$\overline{f}(x) = 0$

Integrally Convex Set



Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt

hold for integrally convex functions

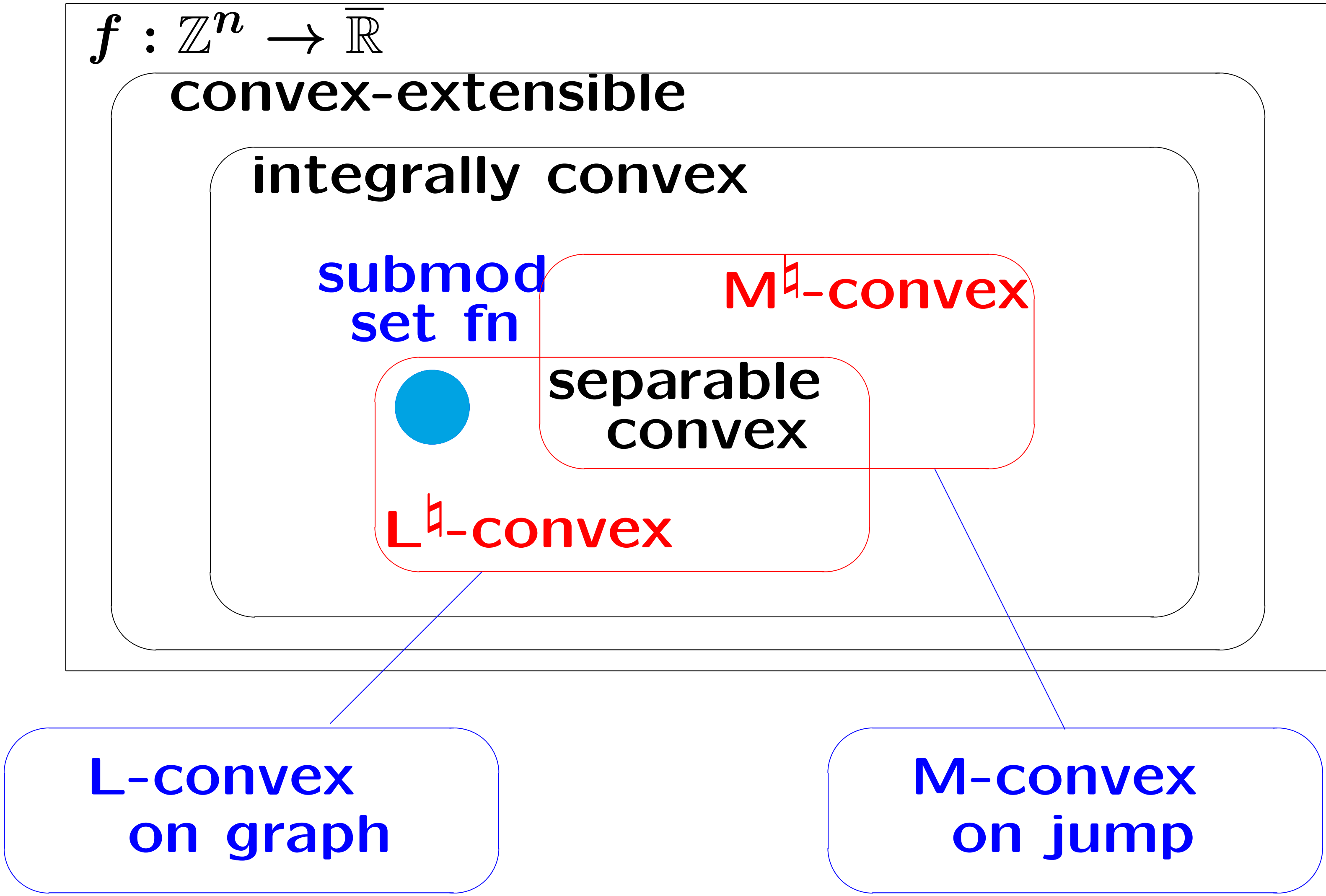
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

fail for integrally convex functions

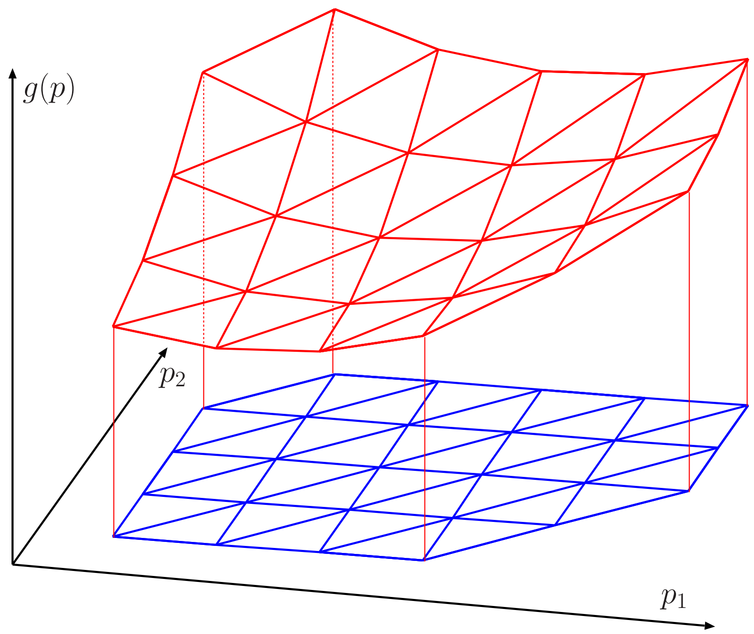
Definitions

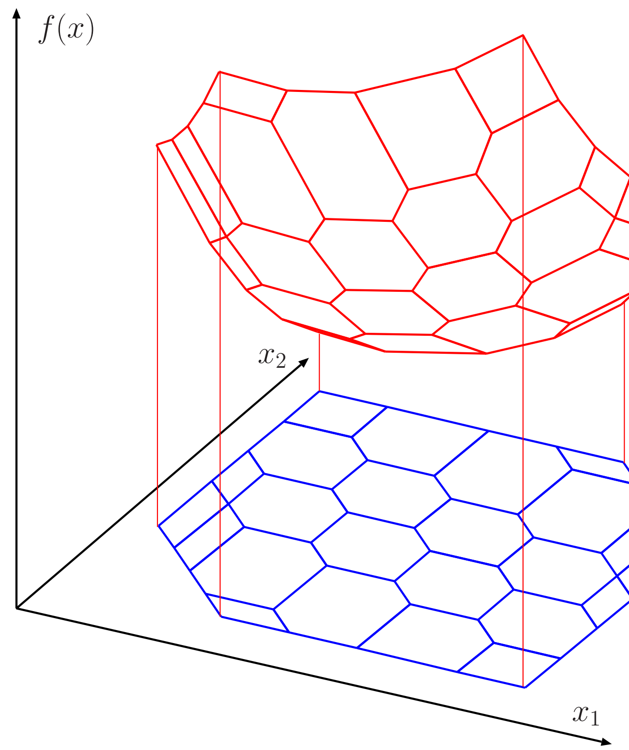
1. submodular (set fn)	$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$
1. separable -conv	$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$ $\varphi_i(t-1) + \varphi_i(t+1) \geq 2\varphi_i(t) \ \ (\forall t \in \mathbb{Z})$
1. integrally -conv	Local convex ext $\tilde{f}(x)$ is convex
2. L-conv($\mathbb{Z}^n$)	
2. M-conv($\mathbb{Z}^n$)	
3. M-conv(jump)	
3. L-conv(graph)	

Classes of Discrete Convex Functions



Bivariate L♮- and M♮-convex Functions





L♮-convex fn **M♮-convex fn**

C3.

L-convex Functions

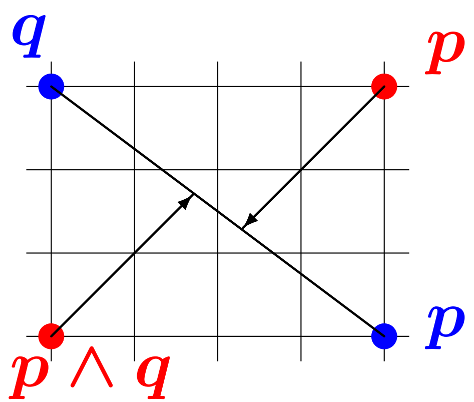
L-convex Function (L = Lattice)

(Murota 98)

$g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

$p \vee q$ compnt-max

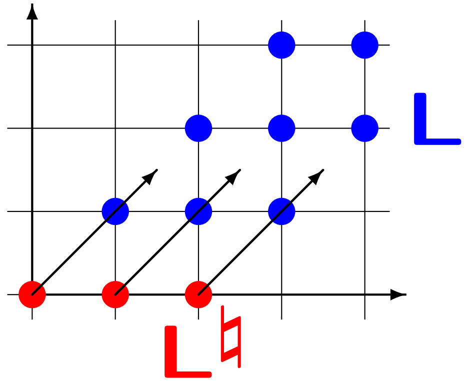
$p \wedge q$ compnt-min



Def: g is L-convex $\iff$

- Submodular: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$
- Translation: $\exists r, \ \forall p$: $g(p + \mathbf{1}) = g(p) + r$

$$\mathbf{1} = (1, 1, \ldots, 1)$$



L$^\natural$-convexity from Submodularity

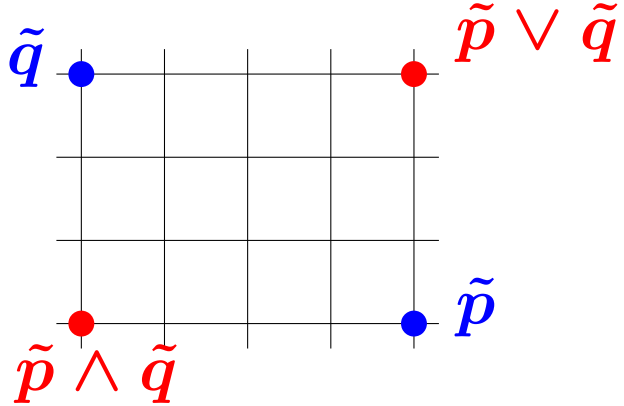
(Murota 98, Fujishige–Murota 2000)

$g : \mathbb{Z}^n \to \mathbb{R}$ **L$^\natural$-convex** $\iff$

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ **is submodular in** (p_0, p)

$\tilde{g} : \mathbb{Z}^{n+1} \to \mathbb{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1)$

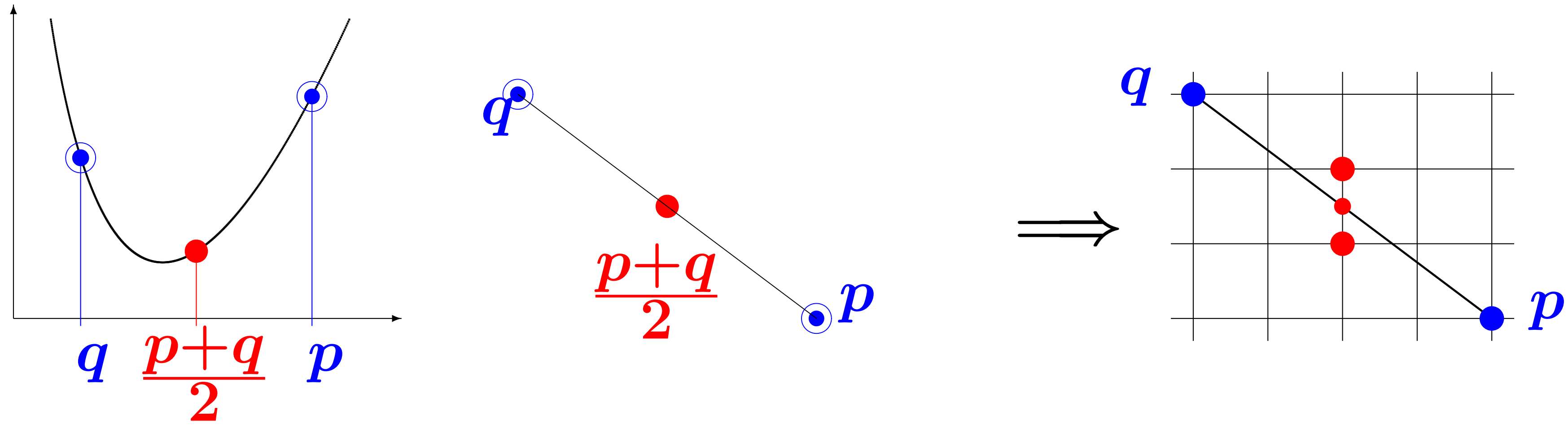
$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}\,(\tilde{p} \vee \tilde{q}) + \tilde{g}\,(\tilde{p} \wedge \tilde{q})$



$\mathsf{L}_{n+1} \simeq \mathsf{L}_n^\natural \supsetneqq \mathsf{L}_n$

L♮-convexity from Mid-pt-convexity

(Favati-Tardella 1990, Fujishige–Murota 2000)



Mid-point convex ($g : \mathbb{R}^n \to \mathbb{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

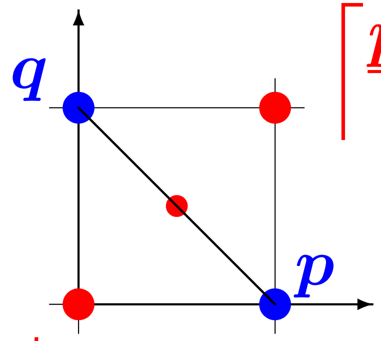
$\Longrightarrow$ **Discrete mid-point convex ($g : \mathbb{Z}^n \to \mathbb{R}$)**

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

L♮-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$



$$\left\lceil \frac{p+q}{2} \right\rceil = p \vee q$$

$$\left\lfloor \frac{p+q}{2} \right\rfloor = p \wedge q$$

Discrete mid-pt convexity:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

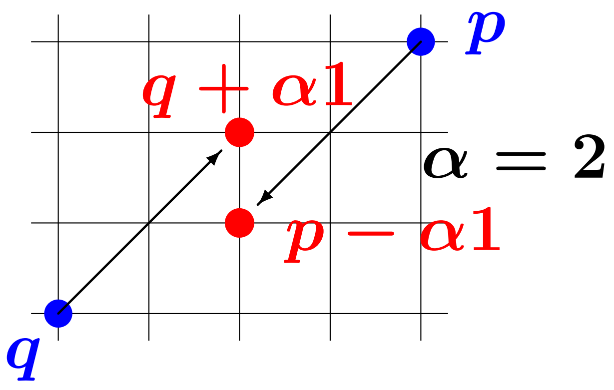
$\iff$ Submodularity:

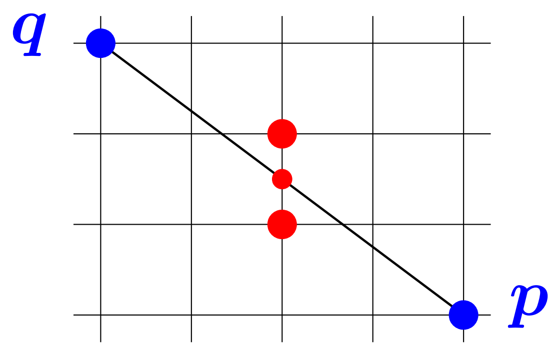
$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

Translation Submodularity ($L^\natural$)

$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1))$$

$(\alpha \geq 0)$





discrete mid-pt convex

$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

(Fujishige-Murota 00)

$\Leftrightarrow$ translation submodular

(Fujishige-Murota 00)

$\Leftrightarrow$ discrete mid-pt convex

(Favati-Tardella 90)

$\Leftrightarrow$ submod. integ. convex

Rem: L♮-convex vs Submodular

$n = 1$

Fact 1: Every $g : \mathbb{Z} \to \mathbb{R}$ is submodular

Fact 2: Function $g : \mathbb{Z} \to \mathbb{R}$ is L♮-convex

$\iff$ $g(p-1) + g(p+1) \geq 2g(p)$ for all $p \in \mathbb{Z}$

L♮-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L♮-convex

$$\Leftrightarrow \ a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Energy function: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathbb{R}}$

$$\Leftrightarrow \ \rho(X) = g(\chi_X) \ \text{ for some L}^\natural\text{-convex } g$$

Multimodular: $h : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is multimodular $\Leftrightarrow$

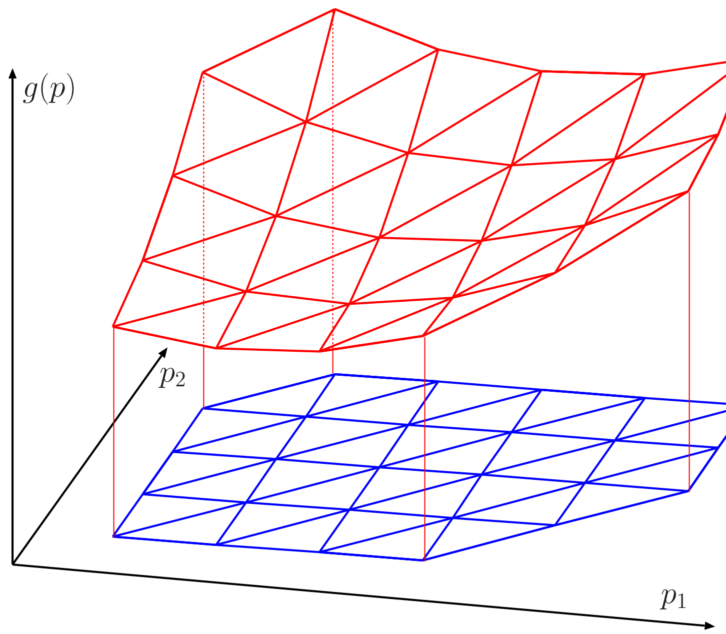
$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L♮-convex g

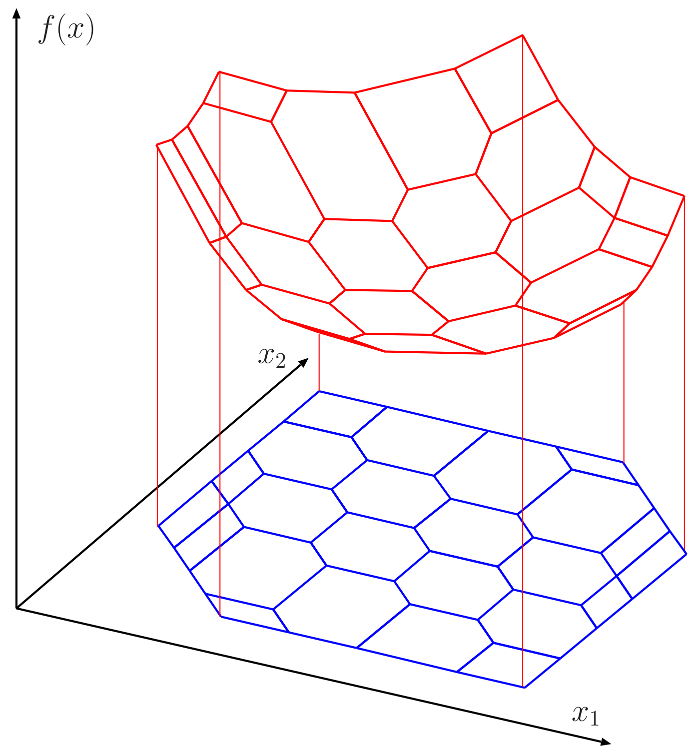
Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

hold for L-convex functions $\Rightarrow$ Part II

Bivariate L♮- and M♮-convex Functions





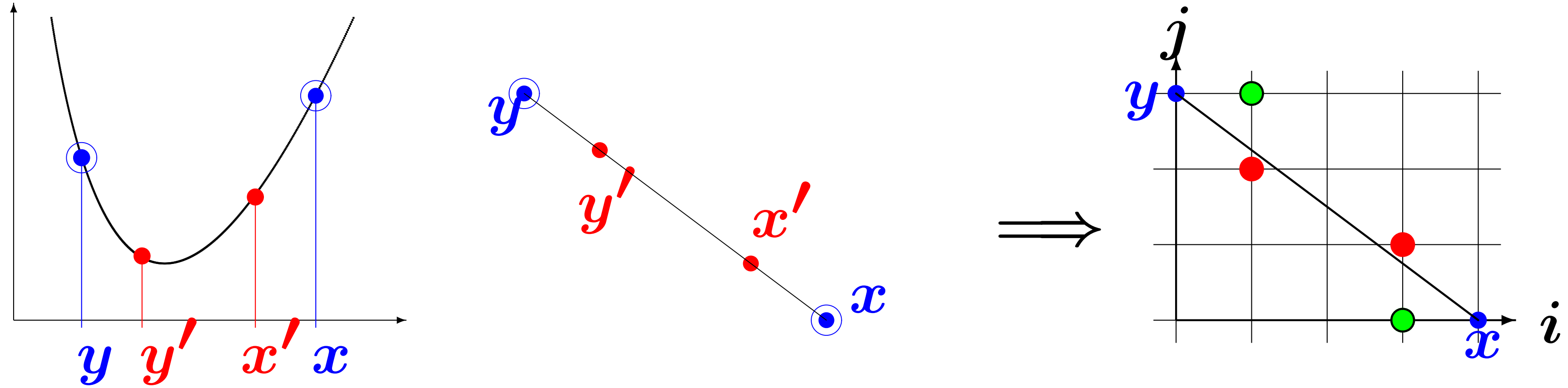
L♮-convex fn **M♮-convex fn**

C4.

M-convex Functions

M♮-convexity from Equi-dist-convexity

(Murota 1996, Murota–Shioura 1999)



Equi-distance convex ($f : \mathbb{R}^n \to \mathbb{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f : \mathbb{Z}^n \to \mathbb{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \Big[f(x - e_i) + f(y + e_i),$$

$$\min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

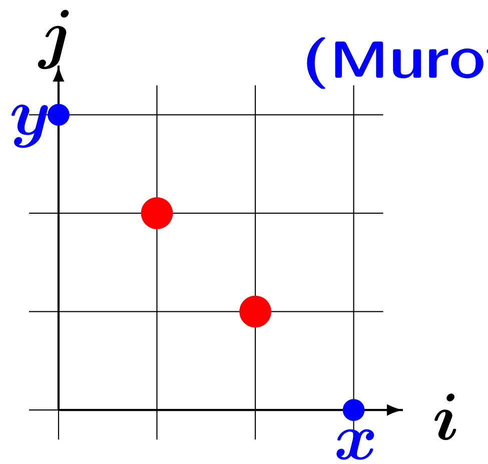
M♮-convex function (M = Matroid)

M-convex Function (M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector

(Murota 96)



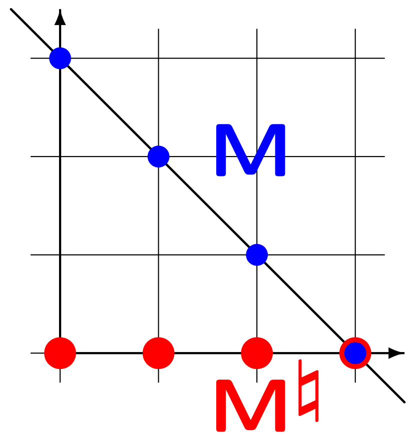
Def: f is M-convex

$\iff \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom}\, f \subseteq$ const-sum hyperplane

$$\mathrm{M}_{n+1} \simeq \mathrm{M}^{\natural}_n \supsetneqq \mathrm{M}_n$$



Gross Substitutes (for set function)

$f : 2^V \to \mathbb{R}$ utility (reservation value) function

p price vector

$D(p) = \arg\max(f - p) = \{X \mid f(X) - p(X)$ is maximum$\}$

demand correspondence

Gross substitutes property: (Kelso–Crawford 82)

$X \in D(p),\ \ p \leq q$

$\Rightarrow\ \ \exists\, Y \in D(q) : \{i \in X \mid p_i = q_i\} \subseteq Y$

Equiv. cond. for $D(p)$ (Gul–Stacchetti 99)

Equiv. cond. for f (Reijnierse–van Gallekom–Potters 02)

& equivalence to M$^\natural$-concavity (Fujishige–Yang 03)

Gross Substitutes for f (not for $D(p)$)

$f : 2^V \to \mathbb{R}$ (set function)

f: gross substitutes $\iff$

(i) $f(S \cup \{i,j\}) + f(S) \leq f(S \cup \{i\}) + f(S \cup \{j\})$

(submodular)

(ii) $f(S \cup \{i,j\}) + f(S \cup \{k\}) \leq$

$\max[f(S \cup \{i,k\}) + f(S \cup \{j\}), f(S \cup \{j,k\}) + f(S \cup \{i\})]$

(Reijnierse–van Gallekom–Potters 02)

cf. Local exchange axiom of M$^\natural$-concave functions

M$\natural$-concavity = Gross Substitutes

$f : \mathbb{Z}^n \to \mathbb{R}$

M$\natural$-concave $\iff$ Gross substitutes $(+**)$

(Reijnierse–van Gallekom–Potters 02, Fujishige-Yang 03
Danilov-Koshevoy-Lang 03, Murota-Tamura 03)

$\Rightarrow$ Shioura-Tamura's survey (J. OR. Soc. Japan, 2015)
https://www.jstage.jst.go.jp/article/jorsj/58/1/58_61/_pdf

Gross substitutes: (f: utility, p: price)

$$x \in \arg\max(f - p), \quad p \leq q,$$

$$\Rightarrow \quad \exists\, y \in \arg\max(f - q) : \; y_i \geq x_i \quad \text{if } p_i = q_i$$

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow\ a_{ij} \geq 0, \qquad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i, j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, ...\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

M♮-concave Functions from Matroids

Matroid rank: $f(X) = r(X)$ (rank of X)

Matroid rank sum: $f(X) = \sum \alpha_i r_i(X)$

$r_i \leftarrow r_{i+1}$ (strong quotient), $\alpha_i \geq 0$ (Shioura 12)

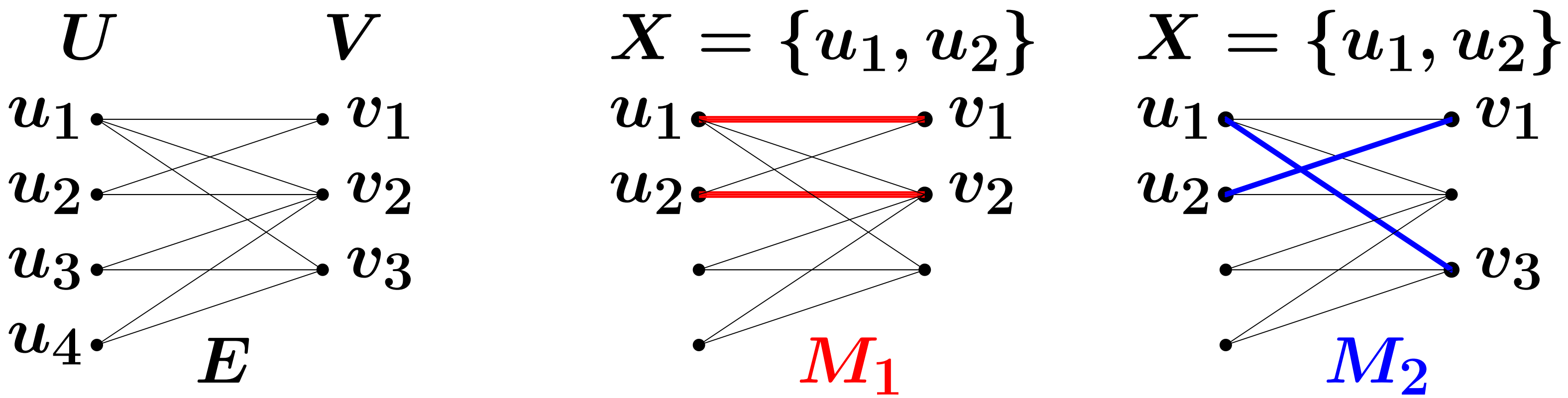
Weighted matroid: w: weight vector

$f(X) = \max\{w(Y) \mid Y\text{: indep} \subseteq X\}$ (Shioura 12)

Valuated matroid: $\omega : 2^V \to \underline{\mathbb{R}}$

$\Leftrightarrow \omega(X) = f(\chi_X)$ for some M-concave f

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight func f is M$^\natural$-concave (cf. Murota 1996)

- Proof by augmenting path

Assignment valuation is GS (cf. Hatfield-Milgrom 2005)

Polynomial Matrix

(Dress-Wenzel 90)
Valuated Matroid

$$A = \begin{array}{|c|c|c|c|} \hline s+1 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \omega(J) = \deg \det A[J]$$

$\mathcal{B} = \{J \mid J$ is a base of column vectors$\}$

Grassmann-Plücker $\Rightarrow$ Exchange (M-concave)

For any $J, J' \in \mathcal{B}$, $i \in J \setminus J'$, there exists $j \in J' \setminus J$
s.t. $J - i + j \in \mathcal{B}$, $J' + i - j \in \mathcal{B}$,

$$\omega(J) + \omega(J') \leq \omega(J - i + j) + \omega(J' + i - j)$$

Ex. $J = \{1, 2\}$, $\quad J' = \{3, 4\}$, $\quad i = 1$

$\det A[\{1,2\}] = \det A[\{3,4\}] = 1$, $\quad \omega(J) = \omega(J') = 0$

Can take $j = 3$: $\quad J - i + j = \{3, 2\}$, $\; J' + i - j = \{1, 4\}$

$$\omega(J - i + j) = 1,\; \omega(J' + i - j) = 1$$

Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
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4. separation theorem
5. Fenchel duality

hold for M-convex functions $\Rightarrow$ **Part II**

C5.

Remarks on Submodular Set Functions

Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

(Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

discrete separation, Fenchel min-max

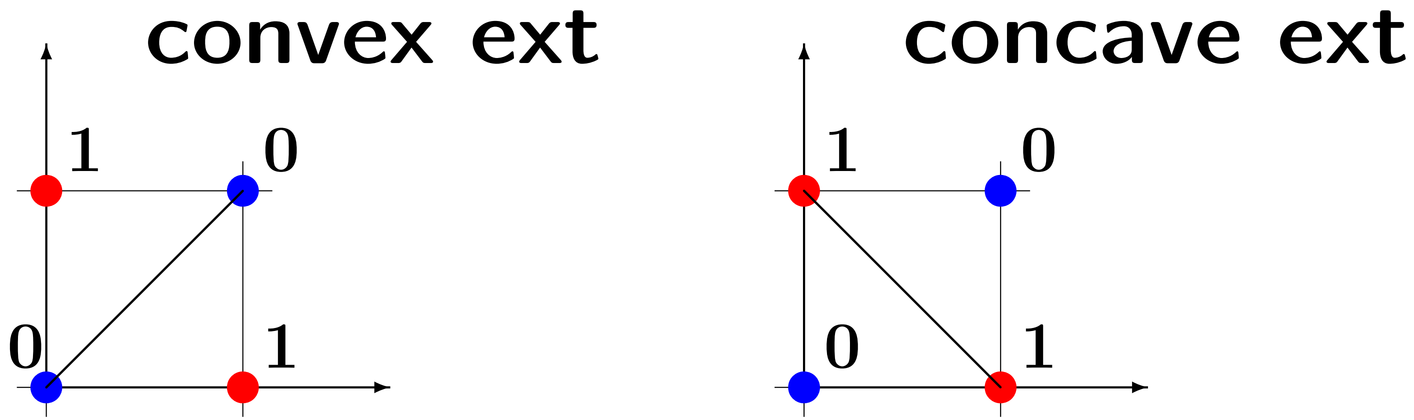
Submodular set functions = Convexity + Discreteness

Set Function and Extensions

Set function $\Longleftrightarrow$ **Function on** $\{0,1\}^n$

$$\rho(X) \quad = \quad \hat{\rho}(\chi_X)$$

Every set function $\rho : \{0,1\}^n \to \mathbb{R}$ can be extended to convex/concave function



cf. Lovász extension

Five Properties of “Convex” Functions

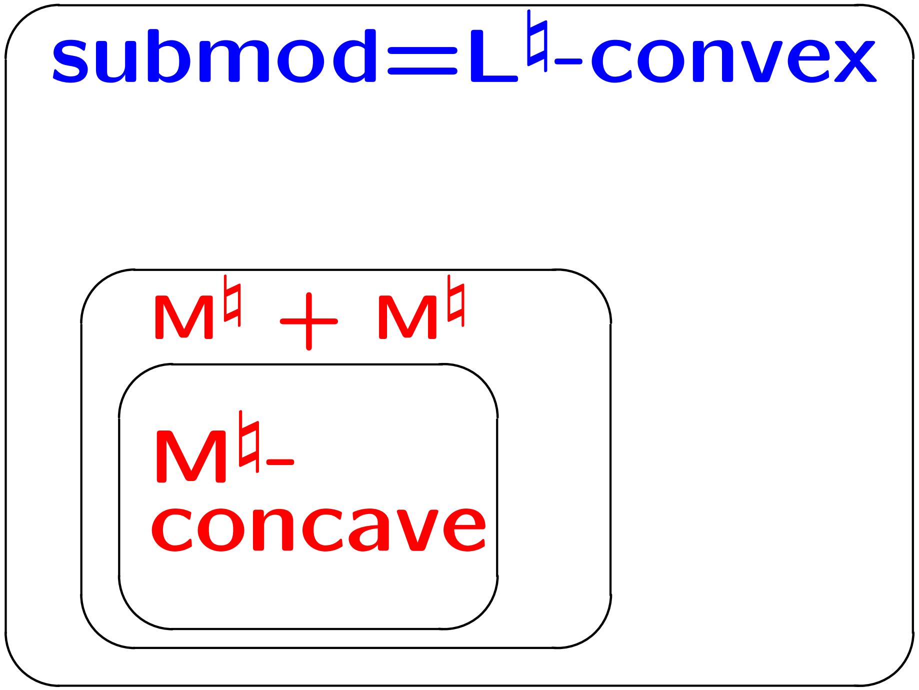
1. convex extension
2. local opt = global opt
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

hold for submodular set functions (1980’s)

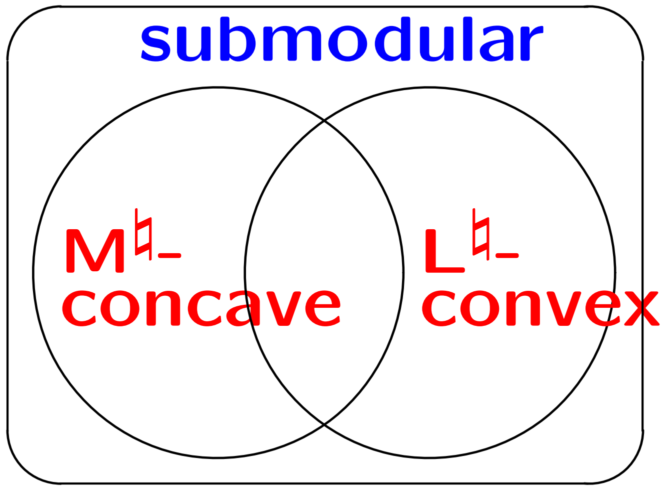
Submodular Set Function in DCA

- Submodular set fn = L♮-convex on $\{0,1\}^n$
- M♮-concave fns form a nice subclass
- M♮ + M♮ are polynomially tractable
- Sums of M♮-concave fns are still OK

$f: \{0,1\}^n \to \mathbb{R}$



$f: \mathbb{Z}^n \to \mathbb{R}$



Dual Character of Matroid Rank Fn

Given a matroid:

$$\rho(X) = \max\{|I| \mid I : \textbf{independent},\ I \subseteq X\}$$

is L$^\natural$-convex and M$^\natural$-concave

Polymatroid rank function is NOT M$^\natural$-concave

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