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Discrete Convex Analysis II:

Properties of Discrete Convex Functions

Kazuo Murota
(Tokyo Metropolitan University)

160510BonnHCMecon2

Features of [Discrete] Convex Functions

- Occurrence in many models

  motivations, applications

- Operations and transformations

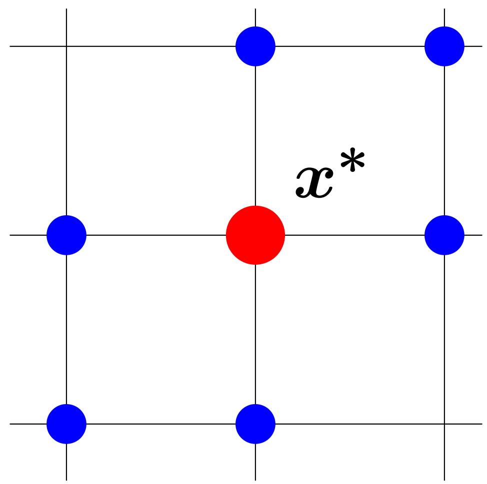
- Sufficient structure for a theory

  mathematically beautiful, practically useful

- Minimization algorithms $\Longrightarrow$ Part III

Local min = Global min

	=?	neighbors	poly-time alg
submodular (set fn)	−		
separable-conv	Y		
integrally-conv	Y		
L-conv ($\mathbb{Z}^n$)	Y		
M-conv ($\mathbb{Z}^n$)	Y		



Contents of Part II

Properties of Discrete Convex Functions

P1. Convex Extension

P2. Operations

P3. Conjugacy

P4. Duality

P1.

Convex Extension

Convex Extension

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is **convex-extensible**

$\Leftrightarrow$ $\exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

Theorem:

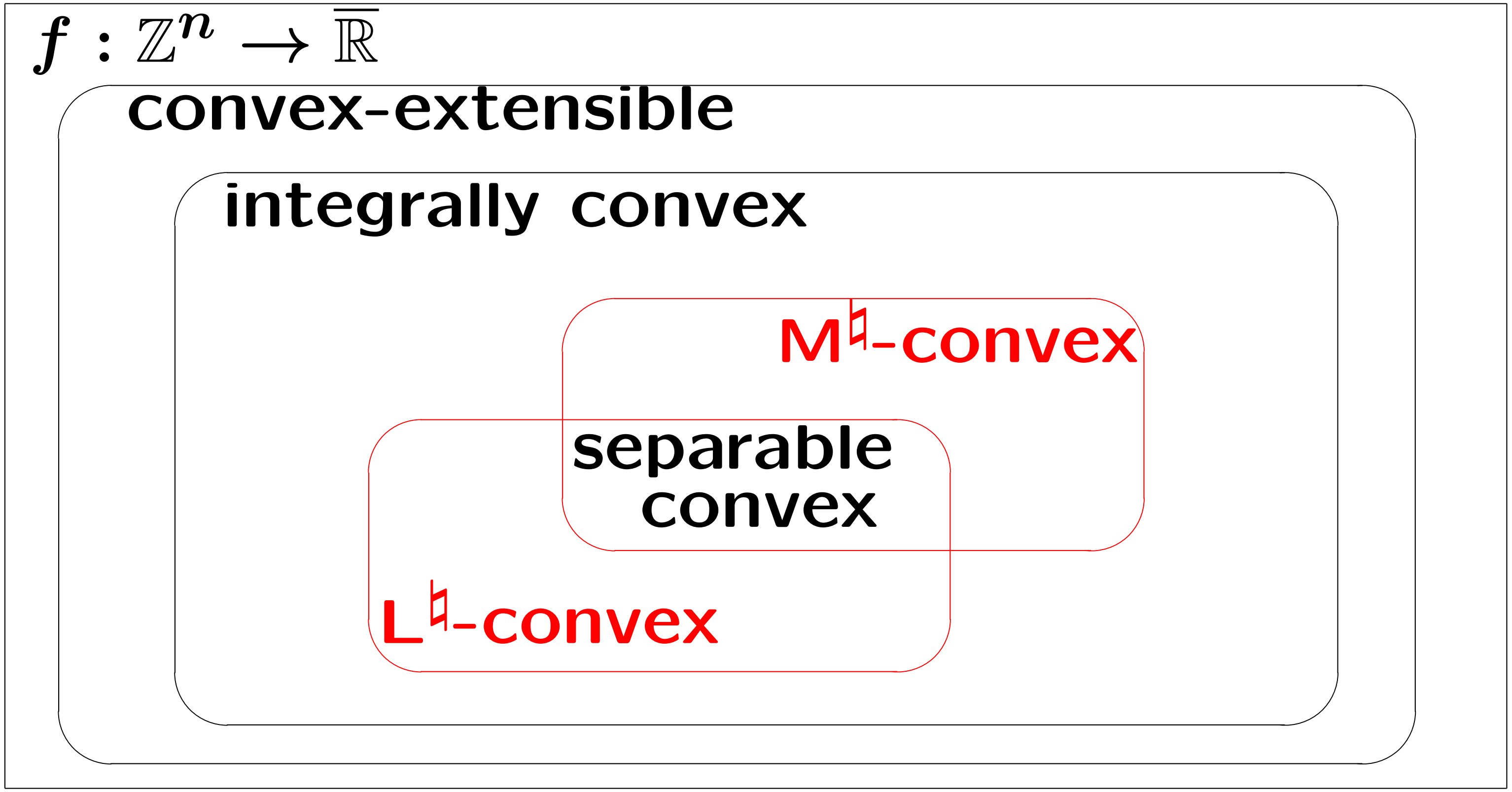
(1) **Separable-convex** fns are convex-extensible

(2) **L$^\natural$-convex** fns are convex-extensible (Murota 98)

(3) **M$^\natural$-convex** fns are convex-extensible (Murota 96)

(4) **Integrally-convex** fns are convex-extensible (by def)

Classes of Discrete Convex Functions



P2.

Operations

Operations

- **scaling:** $af(x) + b$, $f(ax + b)$
- **linear addition:** $f(x) + \langle p, x \rangle$
- **section:** $f(x, 0)$
- **projection:** $\min_{y} f(x, y)$
- **sum:** $f_1(x) + f_2(x)$
- **convolution:** $(f_1 \Box f_2)(x) = \min_{y}(f_1(y) + f_2(x - y))$

- **transformation by graphs/networks**

Scaling/Linear Addition

	$af(x)$ $(a > 0)$	$f(sx)$ $(s \in \mathbb{Z}_+)$	$f(-x)$	$f(x) + \langle p, x \rangle$
submodular (set fn)	Y	—	Y* $V \setminus X$	Y
separable-conv	Y	Y	Y $\pm x_i$	Y
integrally-conv	Y	Y* $(n = 2)$	Y $\pm x_i$	Y
L-conv $(\mathbb{Z}^n)$	Y	Y	Y	Y
M-conv $(\mathbb{Z}^n)$	Y	N	Y	Y

Section/Projection

	section $f(x,0)$	projection $\min_y f(x,y)$
submodular (set fn)	Y restriction	Y contraction
separable-conv	Y	Y
integrally-conv	Y	Y
L-conv ($\mathbb{Z}^n$)	N	Y
$\mathrm{L}^\natural$-conv	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	N
$\mathrm{M}^\natural$-conv	Y	Y

Sum and Convolution

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Theorem: (Murota 98)

f_1, f_2 : L-convex $\Longrightarrow$ $f_1 + f_2$: L-convex

L$^\natural$-convex $\Longrightarrow$ L$^\natural$-convex

$f_1, f_2, \ldots, f_k$: L (L$^\natural$) $\Longrightarrow$ $f_1 + f_2 + \cdots + f_k$: L (L$^\natural$)

- $(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x - y))$

Theorem: (Murota 96)

f_1, f_2 : M-convex $\Longrightarrow$ $f_1 \Box f_2$: M-convex

M$^\natural$-convex $\Longrightarrow$ M$^\natural$-convex

$f_1, f_2, \ldots, f_k$: M (M$^\natural$) $\Longrightarrow$ $f_1 \Box f_2 \Box \cdots \Box f_k$: M (M$^\natural$)

Rem: M $+$ M is not M, L $\Box$ L is not L

Significance of M-convolution Thm

Concave convolution:

$$(U_1 \Box U_2)(x) = \max_y (U_1(y) + U_2(x-y))$$

$U_1, U_2, \ldots, U_k$: gross-substitute (M$^\natural$-concave)

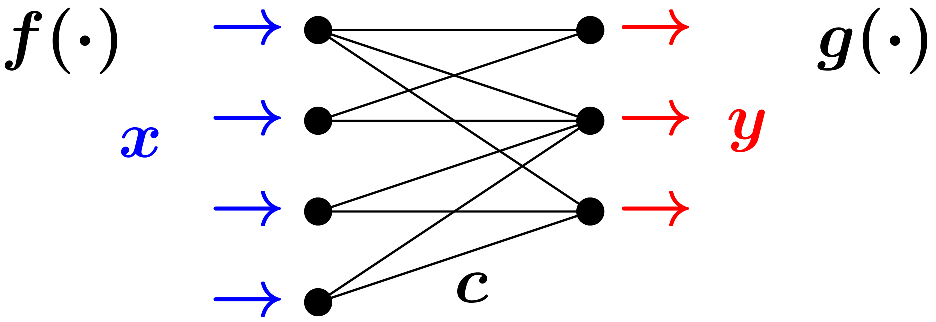
$\Rightarrow$ aggregated utility $U_1 \Box U_2 \Box \cdots \Box U_k$ is gross-substitute (M$^\natural$-concave)

Sum/Convolution

	sum $f_1 + f_2$	convolution $f_1 \Box f_2$
submodular (set fn)	Y	N matroid intersec $\min_{Y \subseteq X} (\rho_1(Y) + \rho_2(X \setminus Y))$
separable-conv	Y	Y
integrally-conv	N	N
L-conv $(\mathbb{Z}^n)$	Y	N $\rightarrow$ L$_2$-convex
M-conv $(\mathbb{Z}^n)$	N $\rightarrow$ M$_2$-conv matr.intersec	Y matroid union

Transformation by Graph/Network

Given
G, f, c



$$\min\{f(x) + c(\xi) \mid x \longleftrightarrow y\} =: g(y)$$

$\uparrow$

$\exists$ flow $\xi : \partial\xi = x \oplus (-y)$

$\exists$ matching $M : \partial M = X \cup Y$

Theorem: (Murota 96)

f : M-convex $\Longrightarrow$ g: M-convex

M$^\natural$-convex $\Longrightarrow$ M$^\natural$-convex

Independent Assignment Valuation

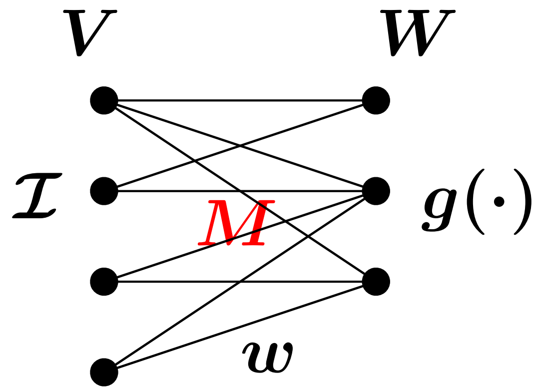
$G = (V, W; E)$: bipartite graph, w: edge weight

$(V, \mathcal{I})$: matroid ($\mathcal{I}$: indep. sets)

Corollary:

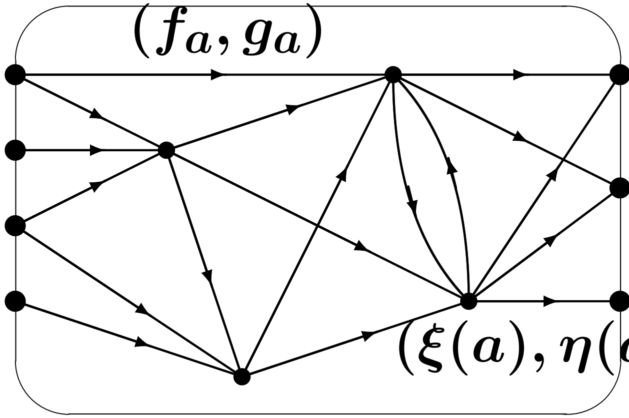
$$g(Y) = \max\{w(M) \mid M\text{: matching}, \quad V \cap \partial M \in \mathcal{I},\ W \cap \partial M = Y\}$$

is M$^\natural$-concave



Transformation by Network

x, y: flow
p, q: potential



M-convex ($\mathbb{Z}^n$): $(y \in \mathbb{Z}^T)$

$$\tilde{f}(y) = \inf_{\xi, x} \{ f(x) + \sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = (x, -y, 0), \quad \xi \in \mathbb{Z}^A, (x, -y, 0) \in \mathbb{Z}^S \times \mathbb{Z}^T \times \mathbb{Z}^{V \setminus (S \cup T)} \}$$

L-convex ($\mathbb{Z}^n$): $(q \in \mathbb{Z}^T)$

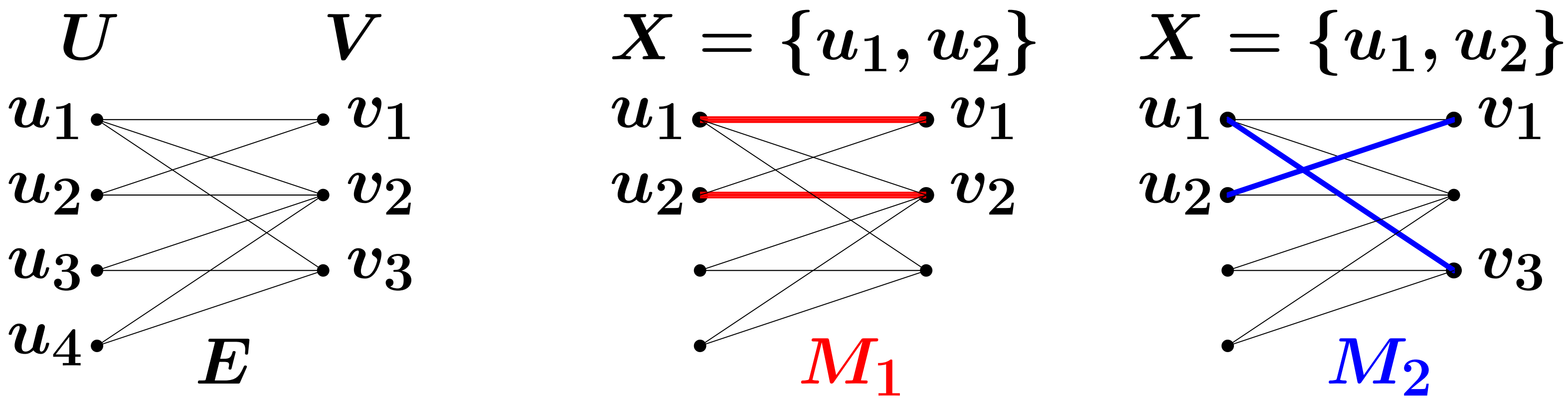
$$\tilde{g}(q) = \inf_{\eta, p, r} \{ g(p) + \sum_{a \in A} g_a(\eta(a)) \mid \eta = -\delta(p, q, r), \quad \eta \in \mathbb{Z}^A, (p, q, r) \in \mathbb{Z}^S \times \mathbb{Z}^T \times \mathbb{Z}^{V \setminus (S \cup T)} \}$$

Transformation by Graphs/Networks

submodular (set fn)	Y (conjugate formula) matroid $\mapsto$ matroid
separable-conv	N
integrally-conv	N
L-conv ($\mathbb{Z}^n$)	Y (conjugate formula)
M-conv ($\mathbb{Z}^n$)	Y

Representation Problem

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X \}$$

Max-weight func f is M$^\natural$-concave (cf. Murota 1996)

- Proof by augmenting path

Assignment valuation is GS (cf. Hatfield-Milgrom 2005)

Endowed Assignment Valuation

Assignment valuation (AV):

$f_w(X)$ = max w-weight of a matching covering X

Endowed assignment valuation (EAV):

$f_{w,T}(X) = f_w(X \cup T) - f_w(T)$

AV $\subseteq$ EAV $\subseteq$ GS (Hatfield-Milgrom 05)

Thm: EAV $\neq$ GS (Ostrovsky-Paes Leme 15)

Proof ideas

- transveral matroids are base orderable (strong exch)
- cycle matroid $\mathcal{M}(K_4)$ is not base orderable
- rank of $\mathcal{M}(K_4) \in \mathrm{GS} \setminus \mathrm{EAV}$

Matroid-based Valuation (MBV)

MBV = (Ostrovsky-Paes Leme 15)

(0) Start with a weighted matroid valuation:

$$f(X) = \max\{w(Y) \mid Y\text{: indep} \subseteq X\}$$

($*$) Repeat

(i) convolution: $\tilde{f}(X) = \max_{Y \subseteq X}(f_1(Y) + f_2(X \setminus Y))$

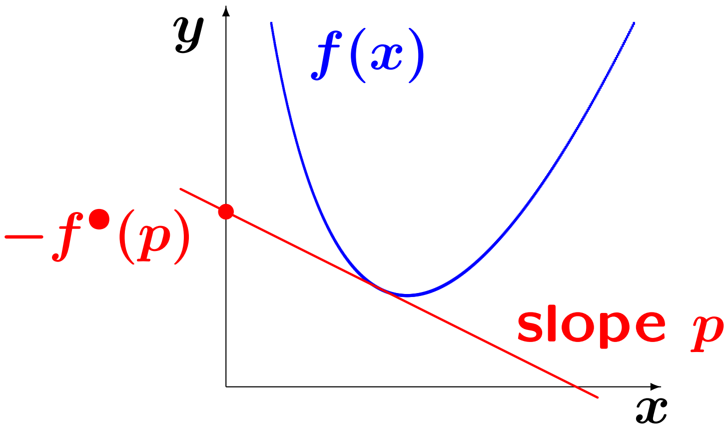
(ii) contraction: $\tilde{f}(X) = f(X \cup T) - f(T)$

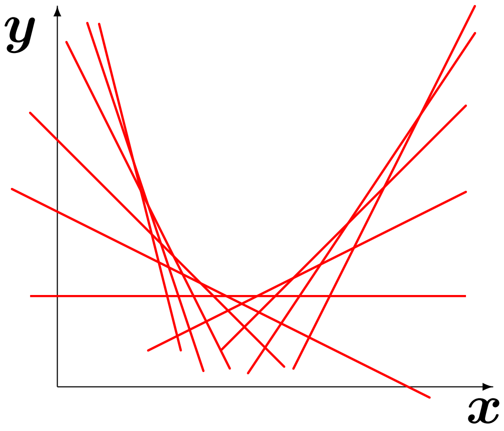
EAV $\subsetneqq$ MBV $\subseteqq$ GS

Conj.: MBV = GS (Ostrovsky-Paes Leme 15)

P3.
Conjugacy

Conjugacy: Discrete Legendre Transform





$$f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} \{\langle p, x \rangle - f(x)\}$$

$\Rightarrow$ **If** $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$, **then** $f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

(integer-valued)

M-L Conjugacy Theorem

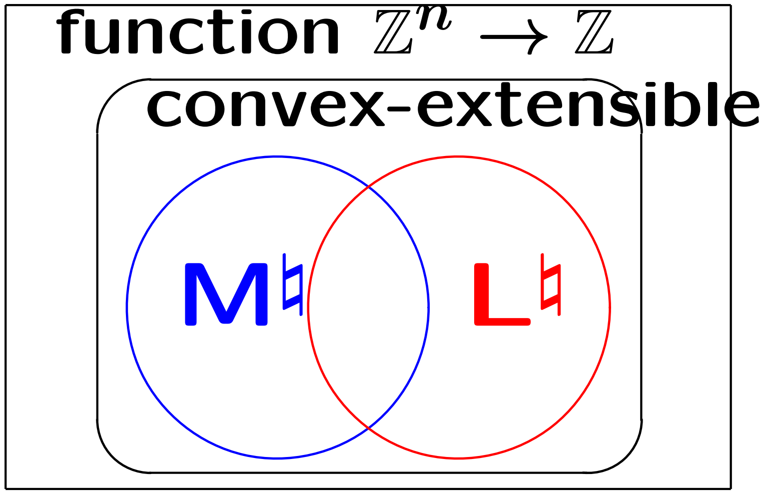
Integer-valued discrete fn $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

(1) M and L are conjugate (Murota 98)

(2) M$^\natural$ and L$^\natural$ are conjugate

$$f \mapsto f^{\bullet} = g \mapsto g^{\bullet} = f$$



(3) biconjugacy

$$f^{\bullet\bullet} = f$$

for $f \in$ M$^\natural$ $\cup$ L$^\natural$

Conjugacy and Biconjugacy

$f:\mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^\bullet:\mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^{\bullet\bullet}=f$
submodular (set fn)	submodular polyhedron $\{x \in \mathbb{Z}^n \mid x(A) \le \rho(A)\}$	Y
separable-convex $f(x)=\sum \varphi_i(x_i)$	separable-convex $\varphi_1^\bullet(p_1)+\cdots+\varphi_n^\bullet(p_n)$	Y
integrally-convex	N	N
L-convex ($\mathbb{Z}^n$)	M-convex	Y
L$^\natural$-convex	M$^\natural$-convex	Y
M-convex ($\mathbb{Z}^n$)	L-convex	Y
M$^\natural$-convex	L$^\natural$-convex	Y

Conjugacy in Polymatroids

Polyhedron S		**Submodular fn ρ**
$S = \{x \mid x(A) \leq \rho(A) \;\; \forall A\}$	$\leftarrow$	
	$\rightarrow$	$\rho(A) = \max_{x \in S} x(A)$

Dual Character of Matroid Rank

$\rho(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$

is L$^\natural$-convex and M$^\natural$-concave

Self-Conjugacy: $\rho(X) = |X| - \rho^\bullet(\chi_X)$

Prf: $\rho^\bullet(\chi_X) = \max_Y\{|X \cap Y| - \rho(Y)\}$

$= \max_{Y \supseteq X}\{|X \cap Y| - \rho(Y)\} = |X| - \rho(X)$

ρ:subm $\Rightarrow$ ρ:L$^\natural$-conv $\Rightarrow$ $\rho^\bullet$:M$^\natural$-conv $\Rightarrow$ ρ:M$^\natural$-concave

History of Discrete Conjugacy

Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	Submodular fn
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid		Lovász extension
Dress–Wenzel 90		Lovász 83
$\Downarrow$		$\Downarrow$
		Submod. integ. conv. fn
		Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	L-convex fn
Murota 96		Murota 98
$\Updownarrow$		$\Updownarrow$
$M^\natural$-convex fn	$\longleftrightarrow$	$L^\natural$-convex fn
Murota -Shioura 99		Fujishige–Murota 00

Significance of Conjugacy

- Economics (auction)
  x: commodity bundle, p: price vector
- Network flow (min-cost flow)
  x: flow, p: tension (potential)
- Electrical network (Iri's book 69)
  x: current, p: voltage (potential)

- Discrete DC programming (Maehara-Murota 15)
  Toland–Singer duality

Implication in Economics

$U(x) : \mathbb{Z}^n \to \mathbb{Z} \cup \{-\infty\}$: utility

$V(p) = \max_x \{U(x) - \langle p, x \rangle\}$: indirect utility

Theorem: (Murota-Shioura-Yang 16)

Utility $U(x)$ is gross-substitute (M$^\natural$-concave)

$\iff$ Indirect utility $V(p)$ is L$^\natural$-convex

Application to Auction:

Ascending auction

$\Rightarrow$ Lyapunov fn min. algorithm (Ausubel 06)

$\Rightarrow$ L$^\natural$-convex fn min. algorithm $\Longrightarrow$ Part III

$\Rightarrow$ Precise analysis (M.-Shioura-Yang 16)

Conjugacy/Duality in Matroids

Conjugacy

Exchange axiom $\Leftrightarrow$ Submodularity of rank function

Duality

Matroid intersection theorem (Edmonds)

Discrete separation (Frank)

Fenchel-type duality (Fujishige)

P4.
Duality

Convex & Concave = Linear

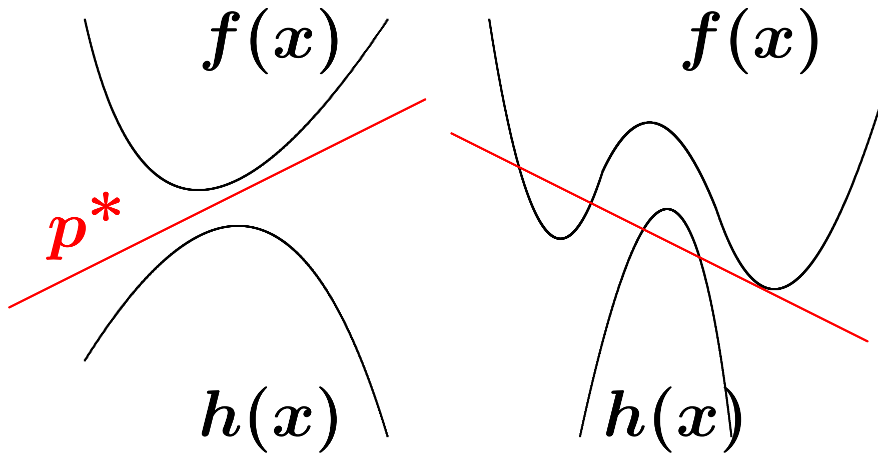
f, $-f$: convex $\Longrightarrow f(x) = \langle p, x \rangle + \alpha$

submodular (set) ($\mathbb{Z}^n$)	Y N
separable-conv	Y
integrally-conv	N (every set fn is int-conv)
L-conv ($\mathbb{Z}^n$)	Y
M-conv ($\mathbb{Z}^n$)	Y

Discrete Separation Theorem

$f : \mathbb{Z}^n \to \mathbb{R}$ "convex"

$h : \mathbb{Z}^n \to \mathbb{R}$ "concave"



- $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbb{R}, \quad \exists\ p^* \in \mathbb{R}^n$:

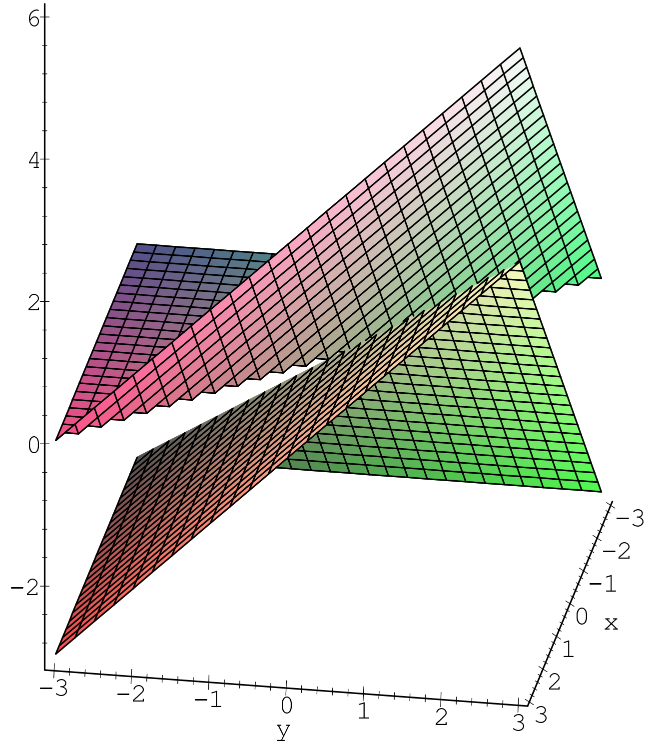
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \quad p^* \in \mathbb{Z}^n$

Difficulty of Discrete Separation (1)

$f(x, y) = \max(0, x + y)$ **convex**

$h(x, y) = \min(x, y)$ **concave**



nonintegral separation

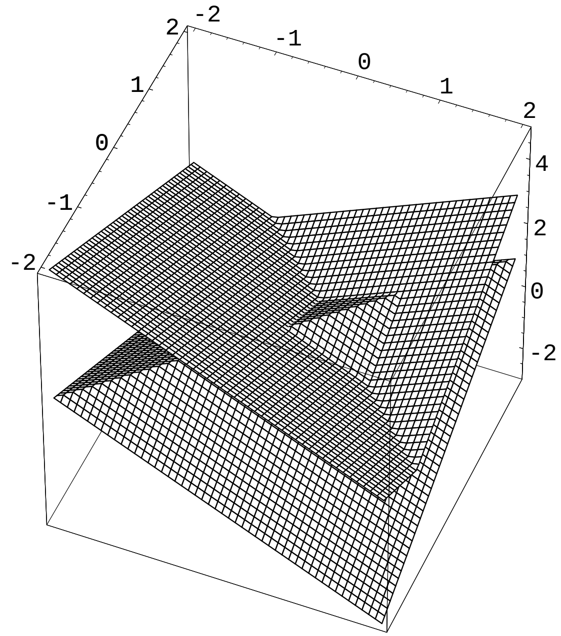
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x,y) = |x + y - 1|$ **convex**

$h(x,y) = 1 - |x - y|$ **concave**



- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathbb{Z}^2)$ **true**
- **No** $\alpha^* \in \mathbb{R}$, $p^* \in \mathbb{R}^2$: $f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$

$\because$ $f = 0 < h = 1$ **at** $(x,y) = (1/2, 1/2)$

Frank's Discrete Separation

(Frank 82)

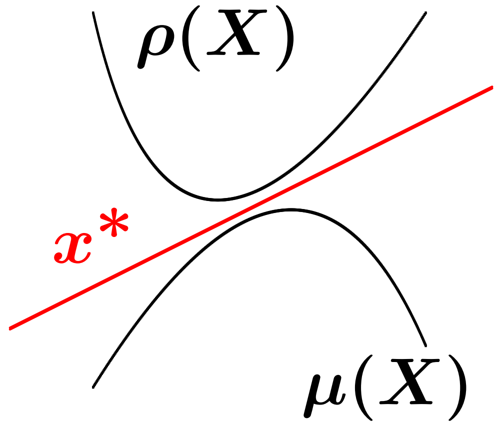
$\rho : 2^V \to \mathbb{R}$: **submodular** $\qquad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathbb{R}$: **supermodular** $\qquad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \exists\ x^* \in \mathbb{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

- $\rho,\ \mu$: **integer-valued** $\Rightarrow x^* \in \mathbb{Z}^V$



Discrete Separation Theorems

(Murota 96/98)

M-separation Thm (for M$^\natural$-convex)

$\Rightarrow$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn $=$ M$^\natural$-convex fn)

L-separation Thm (for L$^\natural$-convex)

$\Rightarrow$ Discrete separ. for submod. set fn (Frank 82)

(submod. set fn $=$ L$^\natural$-convex fn on 0–1 vectors)

Separation and Min-Max Theorems

	separation	min-max
submodular (set fn)	Y (Frank)	Y (Edmonds, Fujishige)
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	Y

Min-Max Duality

f: M$^\natural$-convex, h: M$^\natural$-concave ($\mathbb{Z}^n \to \overline{\mathbb{Z}}$)

Legendre–Fenchel transform

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbb{Z}^n\}$$

Fenchel-type duality thm (Murota 96, 98)

$$\inf_{x \in \mathbb{Z}^n}\{f(x) - h(x)\} = \sup_{p \in \mathbb{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

self-conjugate ($f^\bullet$: L$^\natural$-convex, h°: L$^\natural$-concave)

$\Longrightarrow$ Edmonds' intersection thm
Fujishige's Fenchel duality thm

Edmonds' Intersection Theorem

Submodular polyhedron $(\rho(\emptyset) = 0, \ \rho(V) < +\infty)$

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^n \mid x(X) \leq \rho(X) \ (\forall X \subseteq V)\} \qquad (|V| = n)$$

Theorem: (Edmonds 70)

(1) For $\rho_1, \rho_2 : 2^V \to \overline{\mathbb{R}}$**: submodular,**

$$\max_x \{x(V) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2)\} = \min_X \{\rho_1(X) + \rho_2(V \setminus X)\}.$$

(2) If ρ_1 **and** ρ_2 **are integer-valued, then**

$$\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) = \overline{\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) \cap \mathbb{Z}^n}$$

and there exists $x^* \in \mathbb{Z}^n$ **that attains the maximum.**

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation	
$f(x) \geq \boxed{\text{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84) matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\}$ $= \sup\{h^{\circ} - f^{\bullet}\}$	$\Rightarrow$ discrete separ. for submod (Frank 82) $\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation	
$f^{\bullet}(p) \geq \boxed{\text{Lin}} \geq h^{\circ}(p)$	weighted matroid intersect. (Edmonds 79, Iri-Tomizawa 76, Frank 81)

DCA View on Matroid Union Formula

$\rho(X) = \max\{|I| \mid I :$ **independent**, $I \subseteq X\}$

is L$^\natural$-convex and M$^\natural$-concave

Self-Conjugacy: $\rho(X) = |X| - \rho^\bullet(\chi_X)$

Edmonds' matroid union formula:

$$\max_X \{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y \{\rho_1(Y) + \rho_2(Y) + |V \setminus Y|\}$$

submod maximization **submod minimization**

(M$^\natural$-concave + M$^\natural$-concave) (L$^\natural$-convex + L$^\natural$-convex)

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	—	Y	N	Y*	—	—	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	–	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y

Five Properties of “Convex” Functions

1. convex extension
2. local opt $=$ global opt
3. Legendre transform (biconjugacy)
4. separation theorem
5. Fenchel duality

hold for

- separable-convex functions
- $\mathrm{L}^\natural$-convex functions
- $\mathrm{M}^\natural$-convex functions

END