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Discrete Convex Analysis III:

Algorithms for Discrete Convex Functions

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Contents of Part III

Algorithms for Discrete Convex Functions

A1.
Minimization (General)

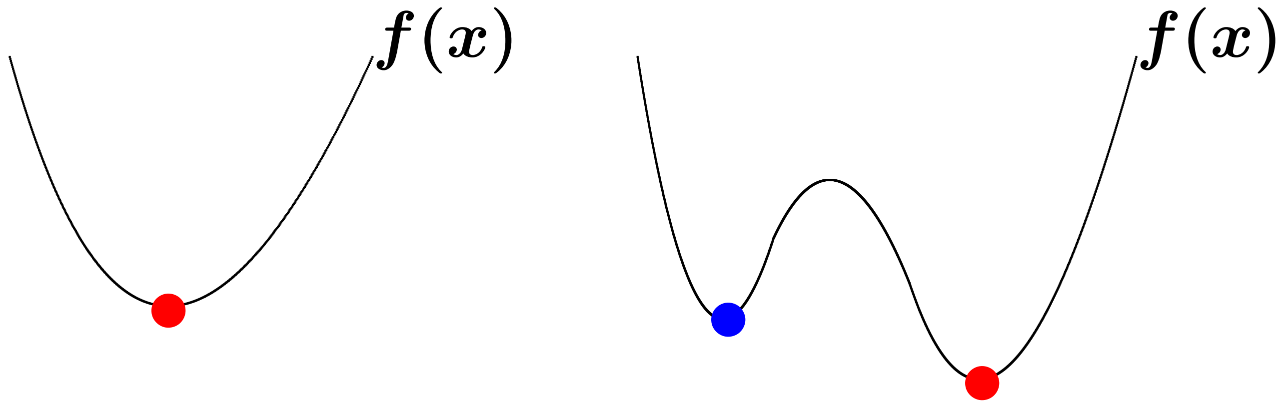
Optimality Criterion

Descent Method

Scaling and Proximity

Optimality Criterion

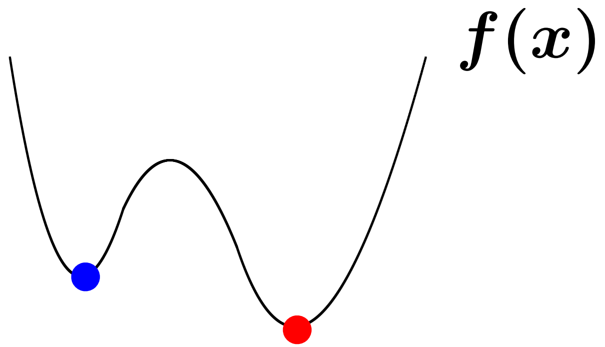
Global opt vs **Local opt**



convex **nonconvex**

Local opt wrt neighborhood

Descent Method



S0: Initial sol x^*

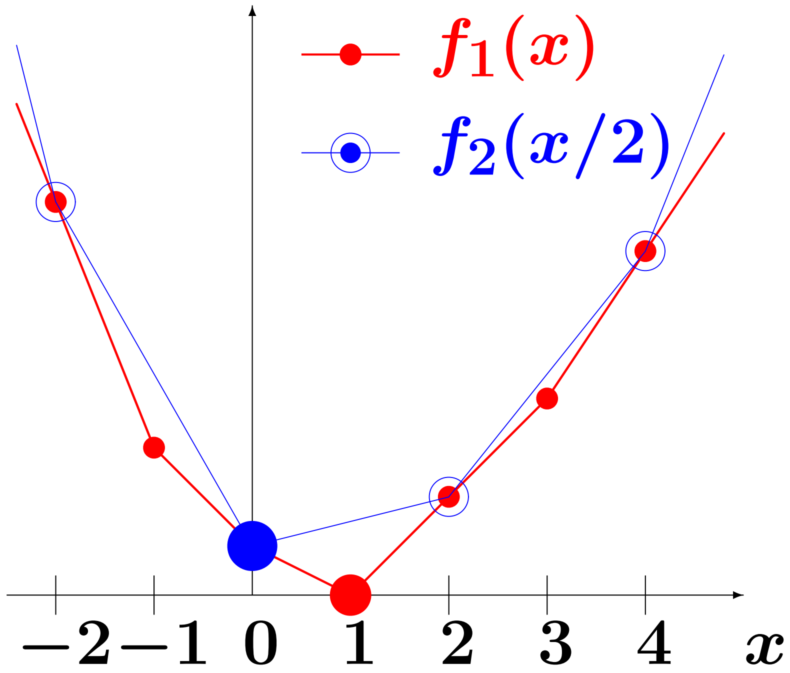
S1: Minimize $f(x)$ in nbhd of x^* to obtain $x^\bullet$

S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (local opt)

S3: Update $x^* := x^\bullet$; go to S1

$\cdots\cdots$What is nbhd ?

Scaling and Proximity



Proximity theorem:

True minimum ● exists

in a neighborhood of

a scaled local minimum ●

$\Rightarrow$ efficient algorithm

Facts in DCA:

- Scaling preserves L-convexity
- Scaling does NOT preserve M-convexity
- Proximity thms known for L-conv and M-conv

A2.

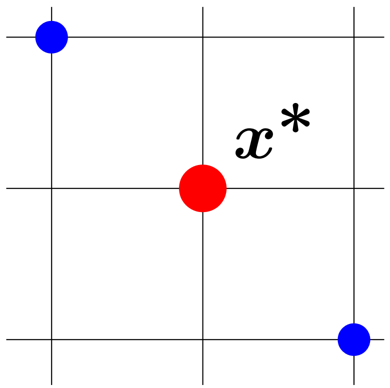
M-convex Minimization

Local vs Global Opt (M-conv)

Thm : $f : \mathbb{Z}^n \to \mathbb{R}$ M-convex (Murota 96)

x^*: global opt

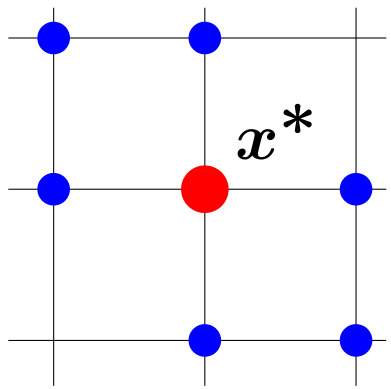
$\iff$ local opt $f(x^*) \leq f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M$^\natural$-convex fn $\Rightarrow$



Steepest Descent for M-convex Fn

S0: Find a vector $x \in \mathrm{dom}\, f$

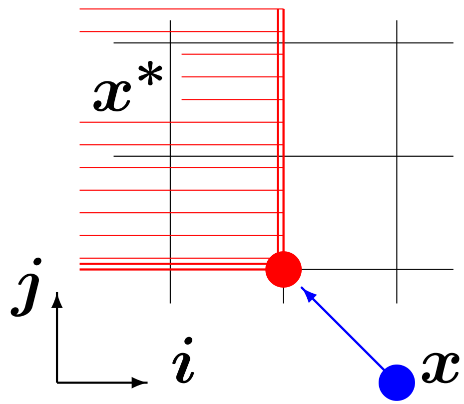
S1: Find (i, j) that minimizes $f(x - e_i + e_j)$

S2: If $f(x) \leq f(x - e_i + e_j)$, stop

(x: minimizer)

S3: Set $x := x - e_i + e_j$

and go to S1



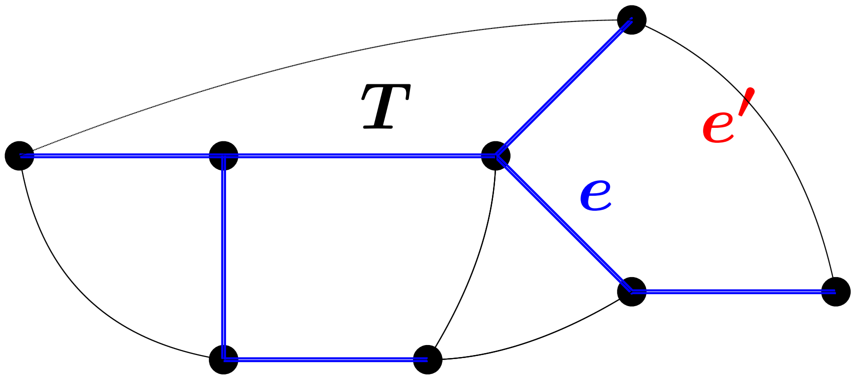
Minimizer Cut Thm (Shioura 98)

$\exists$ minimizer x^* with $x^*(i) \leq x(i) - 1, \ \ x^*(j) \geq x(j) + 1$

$\Rightarrow$ Murota 03, Shioura 98, 03, Tamura 05

- Dress–Wenzel's alg for valuated matroid
- Kalaba's alg for min spanning tree

Min Spanning Tree Problem



length $d: E \to \mathbb{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: **MST** $\iff \tilde{d}(T) \le \tilde{d}(T - e + e')$

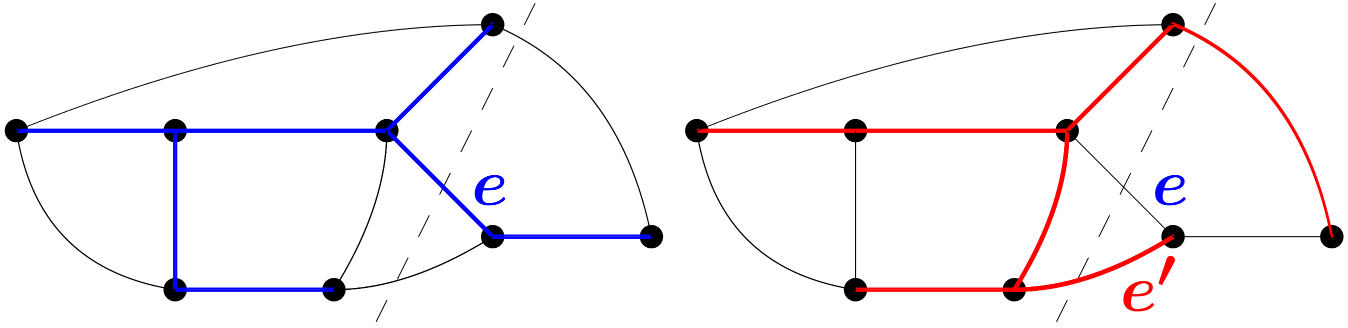
$\iff d(e) \le d(e')$ **if** $T - e + e'$ **is tree**

Algorithm **Kruskal's, Kalaba's**

DCA view

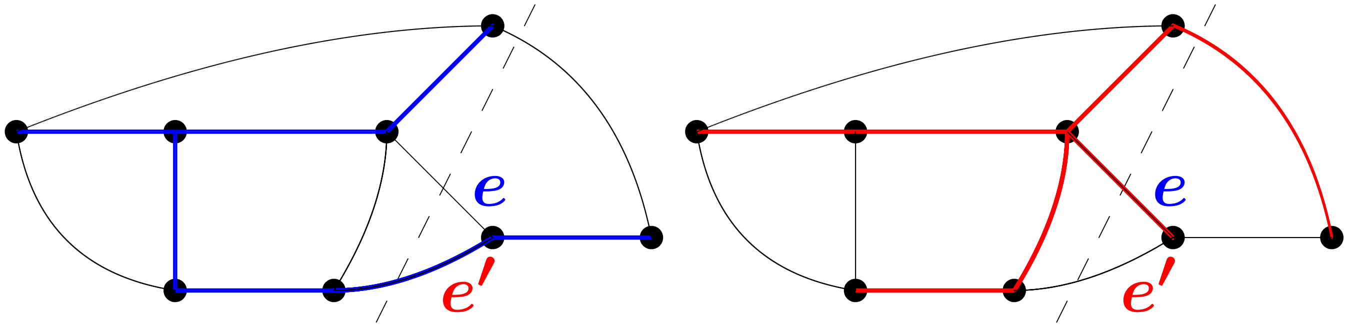
- linear optimization on an M-convex set
- M-optimality: $f(x^*) \le f(x^* - e_i + e_j)$

Tree: Exchange Property



T $\qquad$ T'

Given pair of trees



$T - e + e'$ $\qquad$ $T' + e - e'$

New pair of trees

Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

A3.

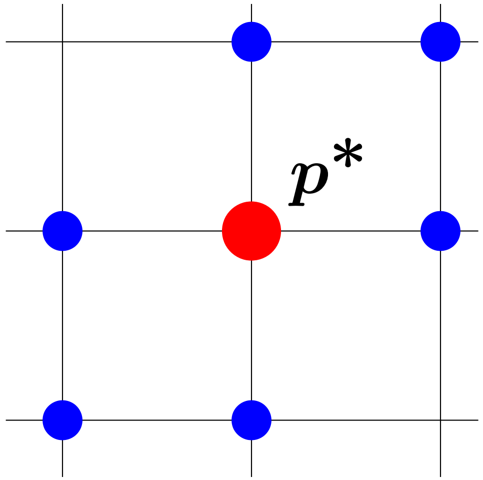
L-convex Minimization

Local vs Global Opt (L♮-conv)

Thm : $g : \mathbb{Z}^n \to \mathbb{R}$ L♮-convex (Murota 98,03)

p^*: global opt

$\iff$ local opt $g(p^*) \leq g(p^* \pm q)$ $(\forall q \in \{0,1\}^n)$



Ex: $p^* + (0,1,0,1,1,1,0,0)$

$\iff$ $\rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$
takes min at $X = \emptyset$

Can check with n^5 (or less) fn evals using submodular fn min algorithm
(Iwata-Fleischer-Fujishige, Schrijver,Orlin, Lee-Sidford-Wong)

Steepest Descent for L$^\natural$-convex Fn

(Murota 00, 03, Kolmogorov-Shioura 09, Murota-Shioura 14)

S0: Find a vector $p^\circ \in \mathrm{dom}\, g$ and set $p := p^\circ$

S1: Find $\varepsilon = \pm 1$ and X that minimize $g(p + \varepsilon \chi_X)$

S2: If $g(p) \leq g(p + \varepsilon \chi_X)$, stop ($p$: minimizer)

S3: Set $p := p + \varepsilon \chi_X$ and go to S1

Thm: (Murota-Shioura 14)

Termination exactly in $\mu(p^\circ) + 1$ iterations, where

$$\mu(p^\circ) = \min\{\|p^* - p^\circ\|_\infty^+ + \|p^* - p^\circ\|_\infty^- \mid p^* \in \arg\min g\}$$

$$\|q\|_\infty^+ = \max_i \max(0, q(i)), \qquad \|q\|_\infty^- = \max_i \max(0, -q(i))$$

Monotone Steepest Descent for L$^\natural$-convex Fn

S0: Find a vector $p^\circ \in \mathrm{dom}\, g$ s.t

$\{q \mid q \geq p^\circ\} \cap \mathrm{argmin}\, g \neq \emptyset$ and set $p := p^\circ$

S1: Find X that minimizes $g(p + \chi_X)$

S2: If $g(p) \leq g(p + \chi_X)$, stop (p: minimizer)

S3: Set $p := p + \chi_X$ and go to S1

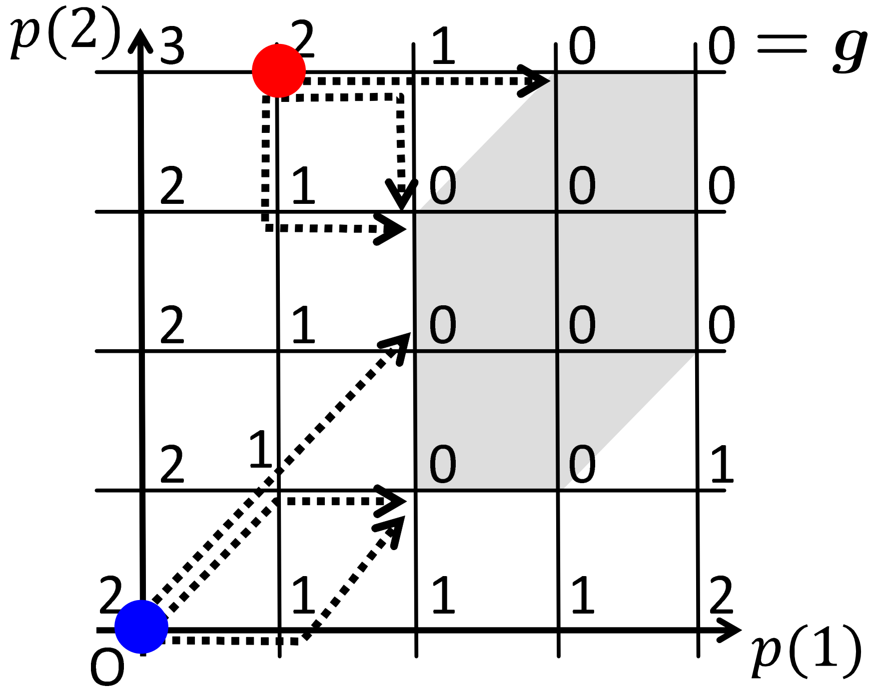
Thm: (Murota-Shioura 14)

Termination exactly in $\hat{\mu}(p^\circ) + 1$ iterations, where

$$\hat{\mu}(p^\circ) = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \arg\min g,\ p^* \geq p^\circ\}$$

$\Rightarrow$ Application to ascending auction

Steepest Descent Path for L$^\natural$-convex Fn



$$\mu(\boldsymbol{p}^\circ) = \hat{\mu}(\boldsymbol{p}^\circ) = 2$$

$$\|\boldsymbol{p}^\circ, \mathrm{argmin}\, g\|_\infty = 1$$

$$\mu(\boldsymbol{p}^\circ) = \hat{\mu}(\boldsymbol{p}^\circ) = 2$$

$$\|\boldsymbol{p}^\circ, \mathrm{argmin}\, g\|_\infty = 2$$

Shortest Path Problem (one-to-all)

one vertex (s) to all vertices, length $\ell \geq 0$, integer

Dual LP

Maximize	$\Sigma\ p(v)$
subject to	$p(v) - p(u) \leq \ell(u,v)\ \forall (u,v)$
	$p(s) = 0$

Algorithm Dijkstra's

DCA view

- linear optimization on an L$^\natural$-convex set (in polyhedral description)
- Dijkstra's algorithm (M.-Shioura 12)
  = steepest ascent for L$^\natural$-concave maximization
  with uniform linear objective $(1, 1, \ldots, 1)$

Optimality & Proximity Theorems

Func Class	Optimality	Proximity
L-convex	$f(x^*) \leq f(x^* + \chi_S) \quad (\forall S)$ $f(x^* + 1) = f(x^*)$ (M. 01)	$\lVert x^* - x^\alpha \rVert \leq (n-1)(\alpha-1)$ (Iwata-Shigeno 03)
M-convex	$f(x^*) \leq f(x^* - \chi_u + \chi_v)$ $(\forall u, v \in V)$ (M. 96)	$\lVert x^* - x^\alpha \rVert \leq (n-1)(\alpha-1)$ (Moriguchi-M.-Shioura 02)
L2-convex (L∗L convol)	$f(x^*) \leq f(x^* + \chi_S) \ (\forall S)$ $f(x^* + 1) = f(x^*)$	$\lVert x^* - x^\alpha \rVert \leq 2(n-1)(\alpha-1)$ (M.-Tamura 04)
M2-convex (M+M)	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W)$ (M. 01)	$\lVert x^* - x^\alpha \rVert \leq \frac{n^2}{2}(\alpha-1)$ (M.-Tamura 04)
integrally convex	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W)$ (Favati-Tardella 90)	???

$\lVert \cdot \rVert = \lVert \cdot \rVert_\infty$

A4.

M-convex Intersection

(Fenchel Duality)

Intersection Problem $(f_1 + f_2)$

Recall: L$^\natural$ + L$^\natural$ $\Rightarrow$ L$^\natural$, M$^\natural$ + M$^\natural$ $\not\Rightarrow$ M$^\natural$

M-convex Intersection Algorithm:

Minimizes $f_1 + f_2$ for M$^\natural$-convex f_1, f_2

$\Leftrightarrow$ Maximizes $f_1 + f_2$ for M$^\natural$-concave f_1, f_2
(submodular function maximization)

$\Leftrightarrow$ Fenchel duality (min = max)

$\Rightarrow$ Valuated matroid intersection (Murota 96)

$\Rightarrow$ Weighted matroid intersection

(Edmonds, Lawler, Iri-Tomizawa 76, Frank 81)

M-convex Intersection: Min [M$^\natural$+M$^\natural$]

M$^\natural$+M$^\natural$ is NOT M$^\natural$

f_1, f_2 : M$^\natural$-convex $(\mathbb{Z}^n \to \mathbb{R})$, $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

(1) x^* minimizes $\boxed{f_1 + f_2}$ (Murota 96)

$\iff$ $\exists\, p$ (certificate of optimality)

- x^* minimizes $\boxed{f_1(x) - \langle p, x\rangle}$ (M-opt thm)
- x^* minimizes $\boxed{f_2(x) + \langle p, x\rangle}$ (M-opt thm)

(2) $\mathrm{argmin}\,(f_1 + f_2) = \mathrm{argmin}\,(f_1 - p) \cap \mathrm{argmin}\,(f_2 + p)$

(3) f_1, f_2 are integer-valued $\Rightarrow$ integral p

M-concave Intersection: Max [M$^\natural$+M$^\natural$]

[Concave version] M$^\natural$+M$^\natural$ is NOT M$^\natural$

f_1, f_2 : M$^\natural$-concave $(\mathbb{Z}^n \to \mathbb{R})$, $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

$f_1 + f_2$ is submodular

(1) x^* maximizes $\boxed{f_1 + f_2}$ (Murota 96)

$\iff$ $\exists\, p$ (certificate of optimality)

- x^* maximizes $\boxed{f_1(x) - \langle p, x \rangle}$ (M-opt thm)
- x^* maximizes $\boxed{f_2(x) + \langle p, x \rangle}$ (M-opt thm)

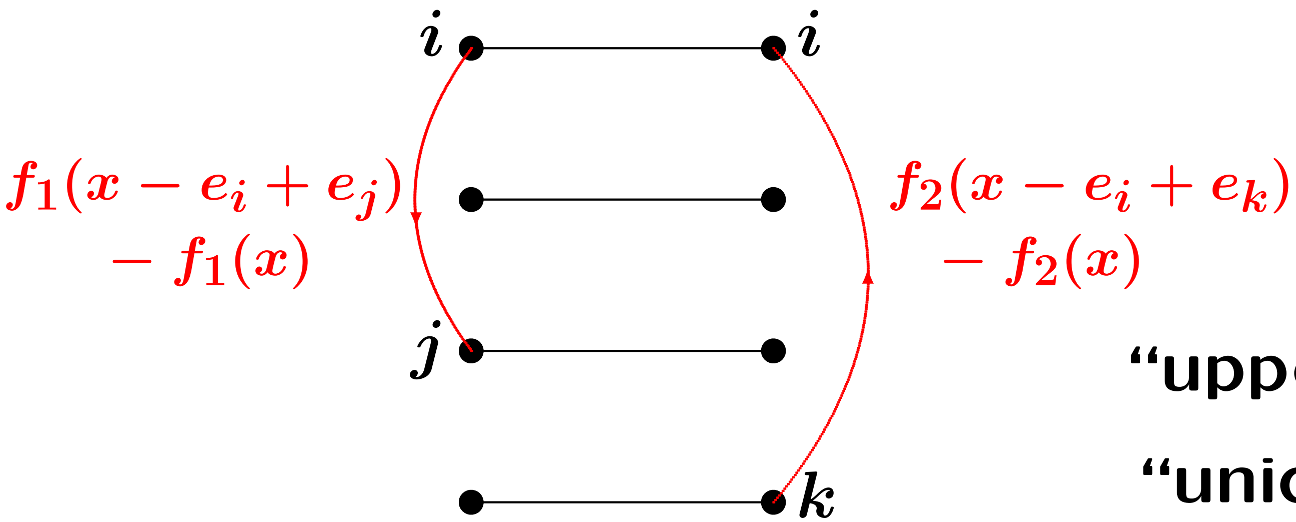
(2) $\mathrm{argmax}\,(f_1 + f_2) = \mathrm{argmax}\,(f_1 - p) \cap \mathrm{argmax}\,(f_2 + p)$

(3) f_1, f_2 are integer-valued $\Rightarrow$ integral p

M-convex Intersection Algorithms

Natural extensions of
weighted (poly)matroid intersection algorithms

Exchange arcs are weighted



"upper-bound lemma"

"unique-max lemma"

- cycle-canceling (Murota 96, 99)
- successive shortest path (Murota-Tamura 03)
- scaling (Iwata-Shigeno 03, Iwata-Moriguchi-M. 05)

Convolution

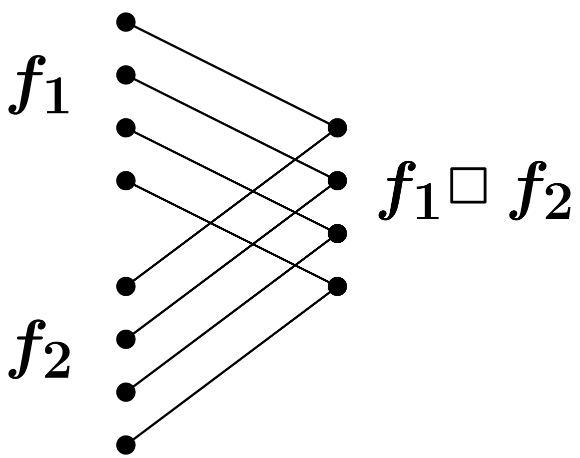
Convolutions of M$^\natural$-convex functions:

$$(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x-y))$$

$$(f_1 \Box f_2 \Box f_3)(x), \qquad (f_1 \Box f_2 \Box \cdots \Box f_k)(x)$$

can be computed by M-convex intersection algorithms

cf. aggregated utility function

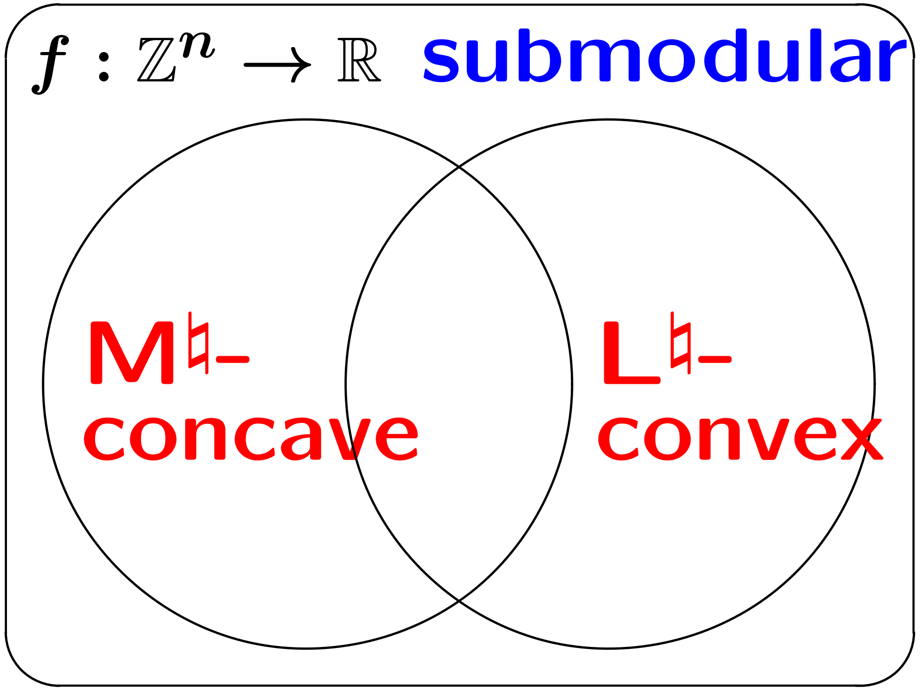


A5.

Submodular Maximization

Submodularity & Convexity in DCA

- M$^\natural$-concave function is submodular
- L$^\natural$-convex function is submodular



sbm + sbm $\Rightarrow$ sbm

L$^\natural$ + L$^\natural$ $\Rightarrow$ L$^\natural$

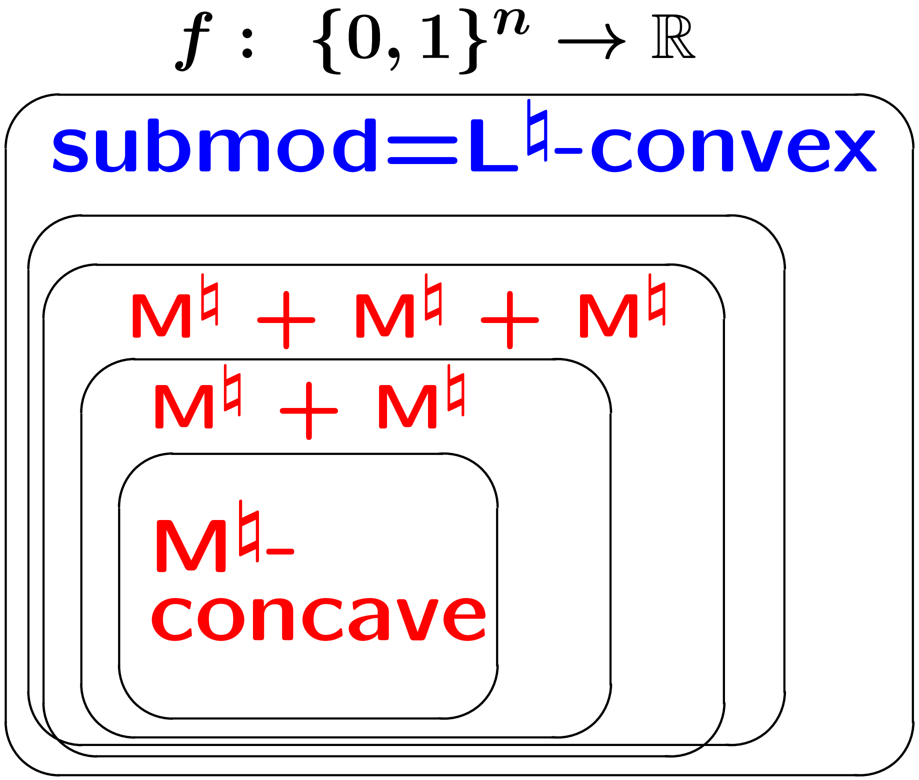
M$^\natural$ + M$^\natural$ $\not\Rightarrow$ M$^\natural$

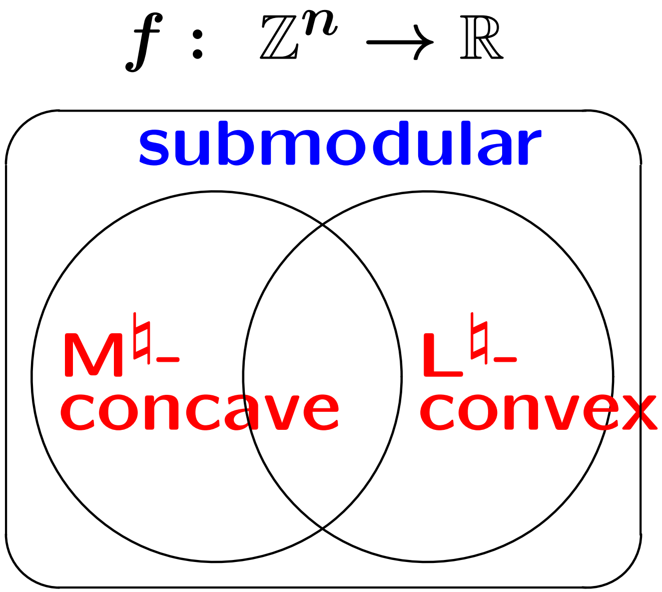
- Sum of M$^\natural$-concave fns is submodular

(i) M$^\natural$ (ii) M$^\natural$ + M$^\natural$ (iii) M$^\natural$ + M$^\natural$ + M$^\natural$

Submodular Set Function in DCA

- **Submodular** set func = L$^\natural$-convex on $\{0,1\}^n$
- (Sums of) M$^\natural$-concave form nice subclasses





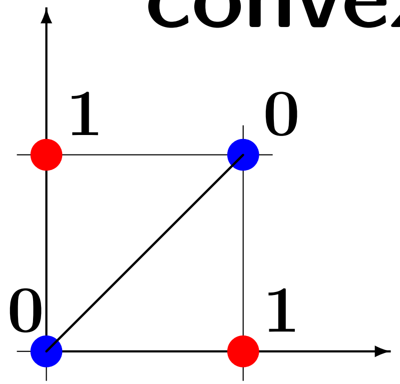
Set Function and Extensions

Set function $\iff$ Function on $\{0,1\}^n$

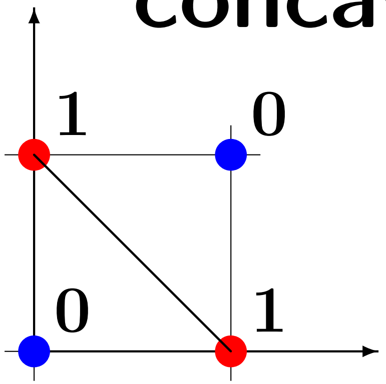
$$\rho(X) = \hat{\rho}(\chi_X)$$

Every set function $\rho : \{0,1\}^n \to \mathbb{R}$ can be extended to convex/concave function

convex ext



concave ext



cf. Lovász extension

M$\natural$-concave Set Functions

M$\natural$-concave is submodular (NOT conversely)

M$\natural$-concave forms a nice subclass for maximization

- $\mu(X) = \varphi(|X|)$ (φ: concave)
- $\mu(X) = \sum_{A \in \mathcal{T}} \varphi_A(|A \cap X|)$ (φ_A: concave)

  $\mathcal{T}$: laminar ($A, B \in \mathcal{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$)
- max-value $\mu(X) = \max\{a_i \mid i \in X\}$
- matroid rank (Fujishige 05)

  $\mu(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$
- weighted matroid rank ($w \geq 0$) (Shioura 09)

  $\mu(X) = \max\{w(I) \mid I : \text{independent}, \ I \subseteq X\}$

Dual Character of Matroid Rank Func

$$\rho(X) = \max\{|I| \mid I : \text{independent}, \ I \subseteq X\}$$

is **L♮-convex** and **M♮-concave**

Self-Conjugacy: $\rho(X) = |X| - \rho^{\bullet}(\chi_X)$

Edmonds' matroid union formula:

$$\max_{X}\{\rho_1(X) + \rho_2(V \setminus X)\} = \min_{Y}\{\rho_1(Y) + \rho_2(Y) + |V \setminus Y|\}$$

submod maximization **submod minimization**

(M♮-concave + M♮-concave) (L♮-convex + L♮-convex)

Algorithms for Submodular Set Func

convex extension	concave extension
computable as Lovász extension (Lovász 03)	subclass: $M^\natural$-concave (valuated matroid)

1. Greedy algorithm for max of an $M^\natural$-concave fn (Dress-Wenzel 90)
2. Matroid intersection-type algorithm for max of a sum of two $M^\natural$-concave fns (Murota 96)
3. Pipage rounding algorithm for approx max of a sum of several nondecr. $M^\natural$-concave fns (Shioura 09)

M$^\natural$-concave Maximization Algorithm

$\mu(X)$: M$^\natural$-concave set function ($\mu(\emptyset) > -\infty$)

Greedy algorithm

S0: Put $X := \emptyset$

S1: Find $j \in V \setminus X$ that $\boxed{\text{maximizes } \mu(X \cup \{j\})}$

S2: If $\mu(X) \geq \mu(X \cup \{j\})$, stop ($X$:maximizer of μ)

S3: Set $X := X \cup \{j\}$ and go to S1

- (Variant of) Dress–Wenzel's for valuated matroid
- Kruskal's algorithm for min spanning tree

Submodular Max. under Matroid Constraint

Maximize $f(S)$ s.t. $|S| = k$

Maximize $f(S)$ s.t. $|S| \leq k$

Maximize $f(S)$ s.t. S: base in a matroid

Maximize $f(S)$ s.t. S: independent in a matroid

Submodular function maximization under a matroid constraint is an NP-hard problem in general

BUT

If f is M$^\natural$-concave, this is an M$^\natural$-concave intersection problem and, hence poly-time solvable

Submodular Welfare Maximization

Maximize $f_1(S_1) + f_2(S_2) + \cdots + f_k(S_k)$ s.t.

$(S_1, S_2, \ldots, S_k)$: partition of S (f_i: submodular)

Submodular welfare maximization is an NP-hard problem in general

BUT

If f_i's are $\mathrm{M}^\natural$-concave, this reduces to an $\mathrm{M}^\natural$-concave intersection (or, $\mathrm{M}^\natural$-concave convolution) and, hence poly-time solvable.

This is the case if $f_i(X) = \varphi_i(|X|)$ with concave φ_i

E N D