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DC Programming in Discrete Convex Analysis

Kazuo Murota

Tokyo Metropolitan University

DC Programming (DC=Difference of Convex)

DC Program: $$\min_{x \in \mathbb{R}^n} \{g(x) - h(x)\}$$

$g, h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, convex

[Tao 1985]
[Tuy 1987]

DC Programming

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Discrete DC Program: $\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$

$g, h: \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$, discrete convex (L♮ or M♮)

[Maehara-Murota 2015]

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[Maehara-Murota 2015]

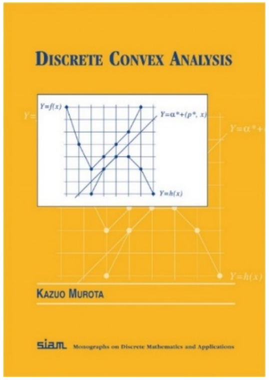
Discrete DC Program: $\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$

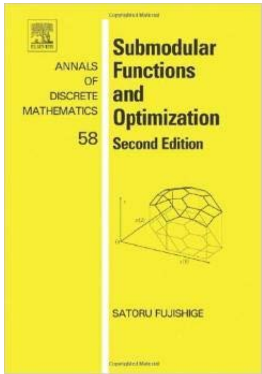
$g, h: \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$, integrally convex

[Murota-Tamura 2018]

Backgrounds

- **DC Programming (DC = Difference-of-Convex)**
  - ➢ DC Algorithm
  - ➢ Subgradient, biconjugacy
- **Discrete Convex Analysis**
  - ➢ Integral convexity
  - ➢ Integral subgradient





1. DC Programming (continuous)
2. Discrete Convex Analysis
3. DC Programming (M/L-convex)
4. DC Programming (integrally convex)

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[Tao 1985]
[Tuy 1987]

Some Features of DC Programming

- **Expressive Power**
  - Almost all optimization problems

- **Mathematical Properties**
  - Toland-Singer duality theorem, Optimality conditions

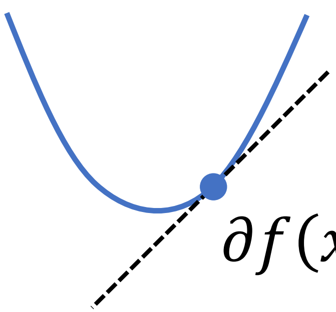
- **Algorithms**
  - Convex analysis approach: for local opt
    subgradient, monotonicity guarantee, fast
  - Combinatorial approach: for global opt
    cutting plane, branch-and-bound

Subgradient & Subdifferential

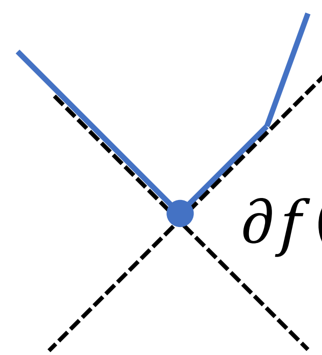
- **Subgradient** p : $f(y) - f(x) \geq \langle p, y - x \rangle \ (\forall y \in \mathbb{R}^n)$

- **Subdifferential** :

$$\partial f(x) = \{ \ p \in \mathbb{R}^n \ \mid \ f(y) - f(x) \geq \langle p, y - x \rangle \ (\forall y \in \mathbb{R}^n) \ \}$$



$\partial f(x) = \{1\}$



$\partial f(x) = [-1, 1]$

DC Algorithm (continuous variables) [Tao c. 1985]

$$\min_{x \in \mathbb{R}^n} \{g(x) - h(x)\}$$

Initial point x

$p \leftarrow$ subgradient of $h(x)$ at x

$h(y) \simeq h(x) + \langle p, y - x \rangle$ (first-order approx)

$x \leftarrow$ solve convex program $\min_{y \in \mathbb{R}^n} \{g(y) - \langle p, y \rangle\}$

Stop if $g(x) - h(x)$ does not decrease

Subdifferentiability & Biconjugacy

$$\partial f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y)\}$$

$$f^{\bullet}(p) = \sup_{x} \ \{\langle p, x \rangle - f(x)\}$$

$$\begin{array}{lcl} p \in \arg\min_{q}\{f^{\bullet}(q) - \langle q, x \rangle\} & \Longleftrightarrow & p \in \partial f(x) \\ & & \Updownarrow \\ x \in \arg\min_{x}\{f(y) - \langle p, y \rangle\} & \Longleftrightarrow & x \in \partial f^{\bullet}(p) \end{array}$$

biconjugacy: $f^{\bullet\bullet} = f$

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2. Discrete Convex Analysis
3. DC Programming (M/L-convex)
4. DC Programming (integrally convex)

Discrete Convex Analysis

[Murota 98]

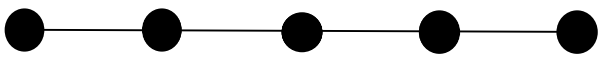
Theory of Matroid & Submodular Func. + Convex Analysis

⇓ ⇓

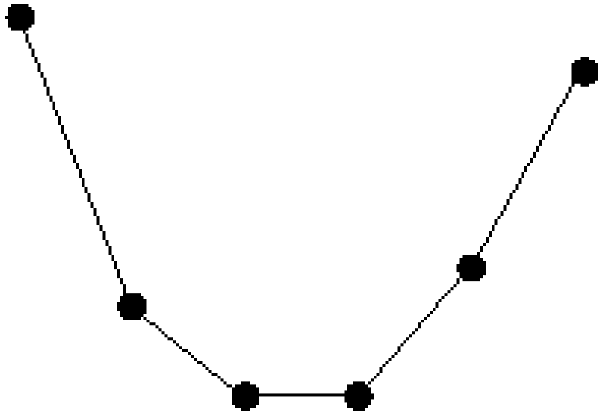
$M^\natural$-convex ⟷ $L^\natural$-convex

$$f : \mathbf{Z}^n \to \mathbf{R}$$

Structures for Discrete Convex Functions (1)



discrete(ly) convex,
separable convex



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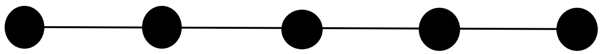

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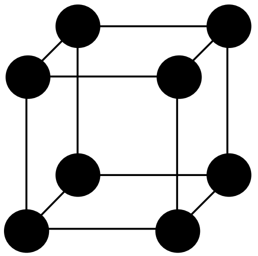
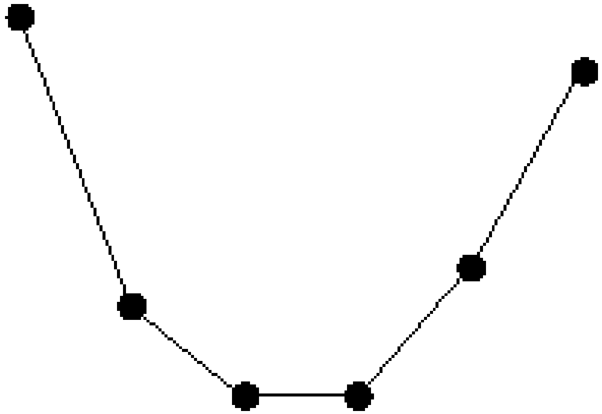


submodular (195*/6*),
valuated matroid (1990)



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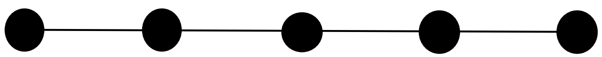

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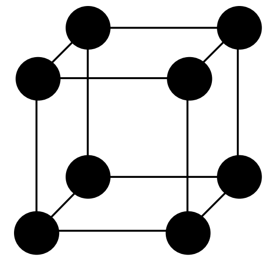

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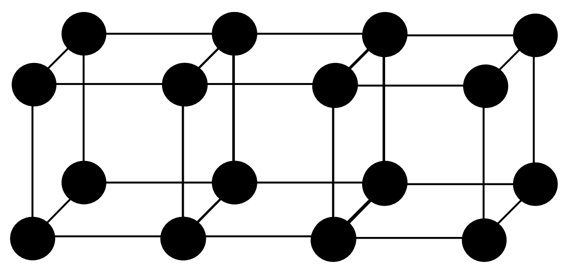
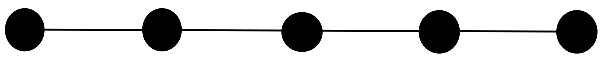


$$f : \mathbf{Z}^n \to \mathbf{R}$$

Structures for Discrete Convex Functions (1)



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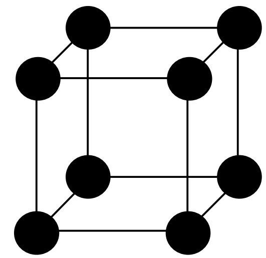

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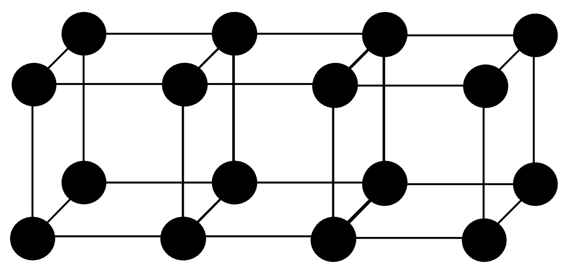
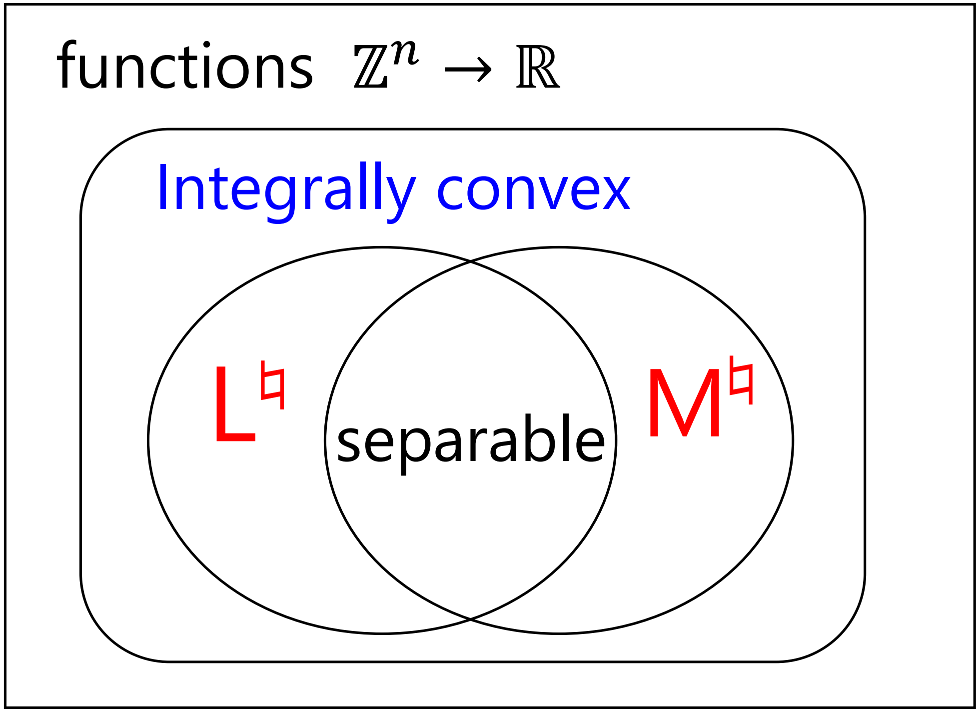
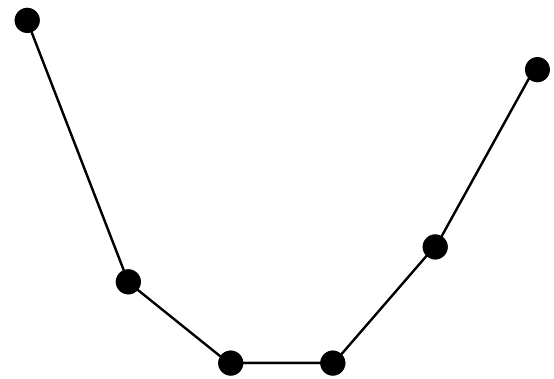


multi-modular (1985), integrally convex (1990),
M-/ $M^\natural$-convex (1996,1999), L-/ $L^\natural$-convex (1998,2000),
BS-conv (2013), UJ-conv (2013)

Discrete Convex Functions



$f\colon \mathbb{Z}^n \to \mathbb{R}$

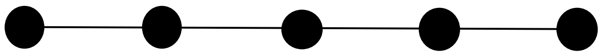


f is separable $\iff$ $f(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n)$

Some History of Discrete Convex Analysis

1935	Matroid	Whitney
1965	Submodular function	Edmonds
1975	Engineering application of matroid	Iri, Recski
1983	Submodularity and convexity	Lovász, Frank, Fujishige
1990	Valuated matroid	Dress–Wenzel
	Integrally convex function	Favati–Tardella
1998	Discrete convex analysis	Murota
2000	Submodular function minimization algorithm	Iwata–Fleischer–Fujishige, Schrijver
2015	Discrete convex functions on graphs	Hirai

Structures for Discrete Convex Functions (1)



discrete(ly) convex,
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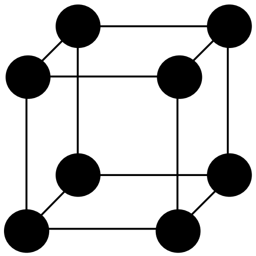
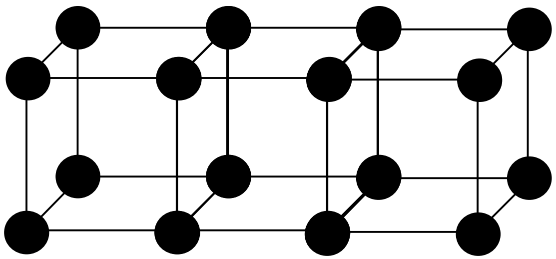


DCA on $\{0,1\}^n$

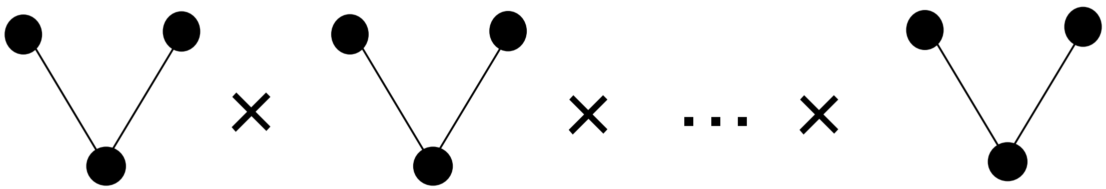
submodular (195*/6*),
valuated matroid (1990)



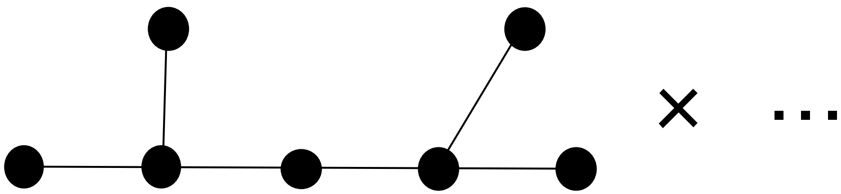
DCA on $\mathbb{Z}^n$

multi-modular (1985), integrally convex (1990),
M-/ M$^\natural$-convex (1996,1999), L-/ L$^\natural$-convex (1998,2000),
BS-conv (2013), UJ-conv (2013)

Structures for Discrete Convex Functions (2)



bisubmodular



L-conv fn on trees/graphs
(Kolmogorov 2011, Hirai 2015)

L-conv fn on CAT(0)

DCA beyond $\mathbb{Z}^n$
(advocated by Hirai)

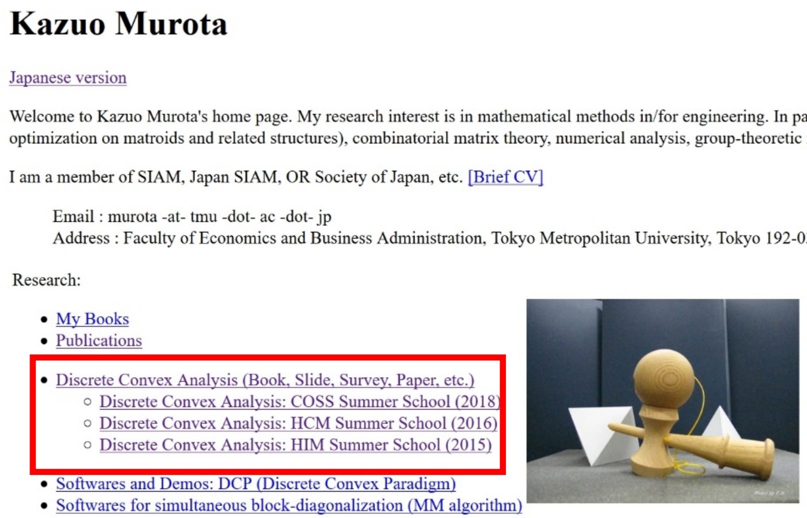
References for Discrete Convex Analysis

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)

Simchi-Levi, Chen, Bramel: The Logic of Logistics, 3rd ed., Springer, 2014 (Chap. 2)

http://www.comp.tmu.ac.jp/kzmurota/index.en.html


Kazuo Murota

Japanese version

Welcome to Kazuo Murota's home page. My research interest is in mathematical methods in/for engineering. In particular, discrete mathematics (combinatorial optimization on matroids and related structures), combinatorial matrix theory, numerical analysis, group-theoretic methods for structural engineering.

I am a member of SIAM, Japan SIAM, OR Society of Japan, etc. [Brief CV]

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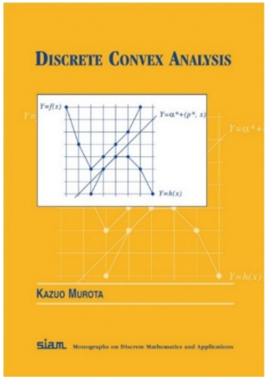
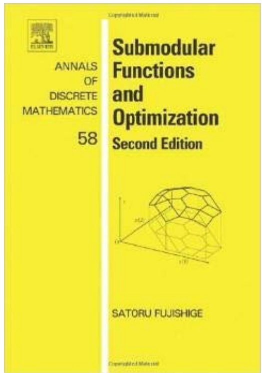
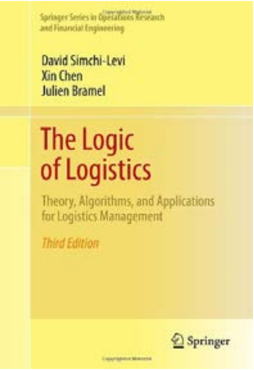
Research:

- My Books
- Publications
- Discrete Convex Analysis (Book, Slide, Survey, Paper, etc.)
  - Discrete Convex Analysis: COSS Summer School (2018)
  - Discrete Convex Analysis: HCM Summer School (2016)
  - Discrete Convex Analysis: HIM Summer School (2015)
- Softwares and Demos: DCP (Discrete Convex Paradigm)
- Softwares for simultaneous block-diagonalization (MM algorithm)







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Discrete DC Programming

$$\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$$

g, h: "convex" (often on 01-vectors)

Some previous works (before 2013)

Quadratic program (Le Thi Hoan An-Pham Dinh Tao 2001)
Discrete tomography (Schüle-Schnörr-Weber-Hornegger 2005)
Mixed Integer program (Niu- Pham Dinh Tao 2008)
Submod-supermod procedure (Narasimhan-Bilmes 2005)
Difference of submod (for global) (Kawahara-Washio 2011)

DC Programming (DC=Difference of Convex)

DC Program: $\min_{x \in \mathbb{R}^n} \{g(x) - h(x)\}$

$g, h: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, convex

[Tao 1985]
[Tuy 1987]

Discrete DC Program: $\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$

$g, h: \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$, discrete convex (L♮ or M♮)

[Maehara-Murota 2015]

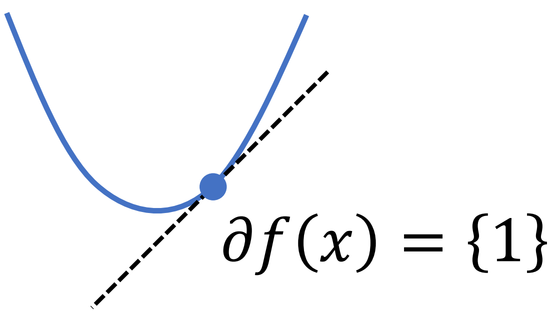
Integral Subgradient & Subdifferential

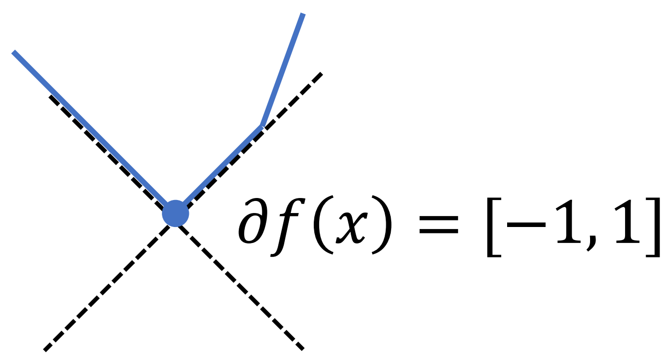
- **Integral subgradient** p :

$$f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y \in \mathbb{Z}^n)$$

- **Integral Subdifferential** :

$$\partial f(x) = \{ \ p \in \mathbb{Z}^n \mid \ f(y) - f(x) \geq \langle p, y - x \rangle \ \ (\forall y \in \mathbb{Z}^n) \ \}$$





[Murota 1998]

Integral Subgradients & Biconjugacy

(Discrete life is not easy)

$f : \mathrm{Z}^n \to \overline{\mathrm{Z}}$ $\qquad$ $\partial_{\mathrm{Z}} f(x) \neq \emptyset$? $\qquad$ $f^{\bullet\bullet} = f$?

Example: $D = \{(0,0,0), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}$

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2 + x_3)/2, & x \in D, \\ +\infty, & \text{o.w.} \end{cases}$$

D is "convex": $\mathrm{conv}(D) \cap \mathrm{Z}^n = D$

$\partial f_{\mathrm{R}}(0) = \{(1/2, 1/2, 1/2)\}$ $\qquad$ $\partial_{\mathrm{Z}} f(0) = \emptyset$

$$f^{\bullet\bullet}(0) = -\inf_{p \in \mathbb{Z}^3} \max\{0, |p_1+p_2-1|, |p_2+p_3-1|, |p_3+p_1-1|\}$$

$f^{\bullet\bullet}(0) = -1 \neq 0 = f(0)$

Subdifferetiability and Biconjugacy

[Murota 1998]

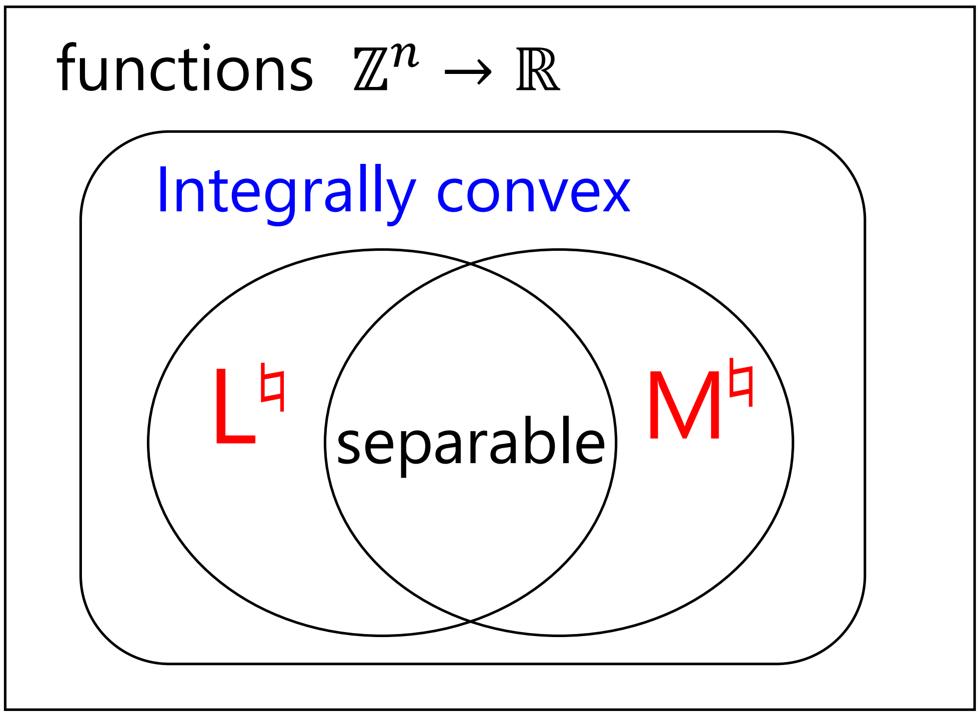
Lemma $f: \mathbb{Z}^n \to \mathbb{Z} \cup \{+\infty\}$: integer-valued

Integral subdifferential $\partial f(x)$ is nonempty
if and only if $f^{\bullet\bullet}(x) = f(x)$

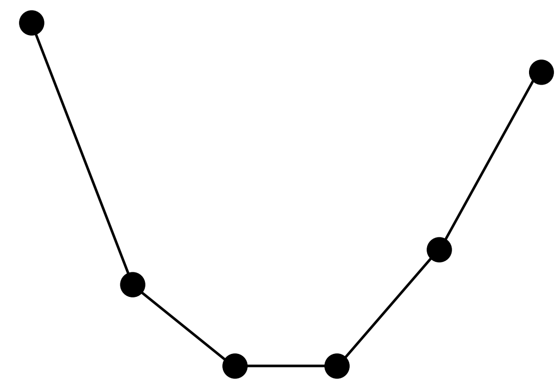
Technical assumption:
Closed convex hull of dom f is rational-polyhedron

Legendre transform: $f^{\bullet}(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

Discrete Convex Functions



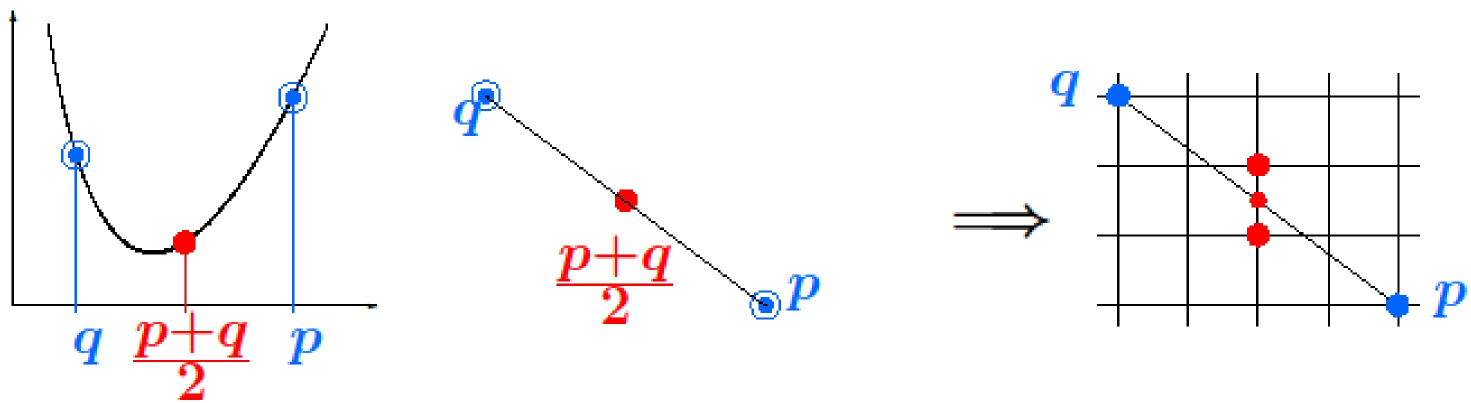
$f\colon \mathbb{Z}^n \to \mathbb{R}$



f is separable $\iff$ $f(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n)$

L♮-convexity from Mid-pt-convexity

(Favati–Tardella 1990, Murota 1998, Fujishige–Murota 2000)



Mid-point convex ($g : \mathbb{R}^n \to \mathbb{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathbb{Z}^n \to \mathbb{R}$)

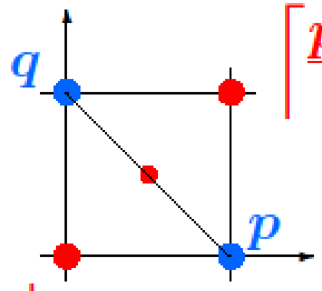
$$g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$

L♮-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$:

$\left\lceil \frac{p+q}{2} \right\rceil = p \vee q$



$\left\lfloor \frac{p+q}{2} \right\rfloor = p \wedge q$

Discrete mid-pt convexity:

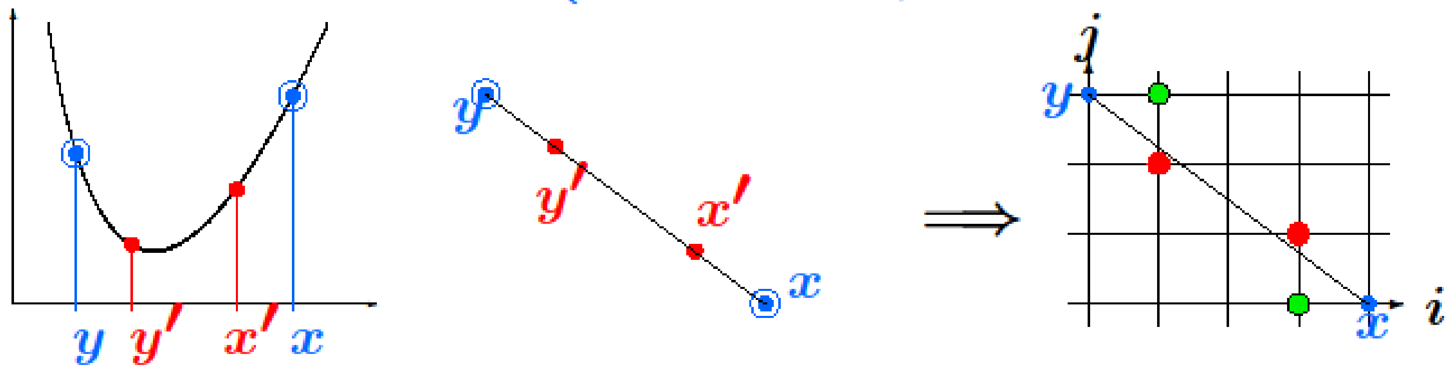
$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

$\Longleftrightarrow$ Submodularity:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

M♮-convexity from Equi-dist-convexity

(Murota 1996, Murota–Shioura 1999)



Equi-distance convex ($f : \mathbf{R}^n \to \mathbf{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange ($f : \mathbf{Z}^n \to \mathbf{R}$)** $\quad \forall x, y, \ \ \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \ \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

M♮-convex function (M = Matroid)

Discrete DC Progr. by Discrete Convex Analysis

[Maehara-Murota 2015]

DC = Difference of Convex

$$\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$$

g, h: "convex" (dom g ⊆ dom h)

Convexity: M♮ −M♮ , M♮ −L♮ , L♮ −M♮ , L♮ −L♮

Examples

- Submod. max. under matroid constraint: M♮ − L♮

max h (x) s.t. x: indep. ⇔ min g(x) - h(x)
h(x): submod, g(x): indicator fn of matroid

- Submod-supermod proc. (Narasimhan-Bilmes): L♮ −L♮

Subdifferetiability and Biconjugacy

[Murota 1998]

Thm f: $\mathrm{M}^{\natural}$- or $\mathrm{L}^{\natural}$- convex, integer-valued

- Subdifferential $\partial f(x)$ is an integral polyhedron ($\neq \emptyset$)
- Hence integral subgradient p exists at every point x
- Hence $f^{\bullet\bullet} = f$

M-L Conjugacy Theorem

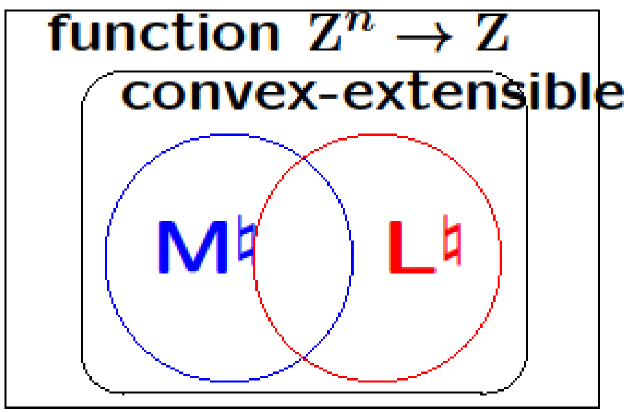
Integer-valued discrete fn $\quad f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $\quad f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

M$^\natural$-convex and L$^\natural$-convex are conjugate

$$f \;\mapsto\; f^{\bullet} = g \;\mapsto\; g^{\bullet} = f$$

(Murota 1998)



biconjugacy

$$f^{\bullet\bullet} = f$$

Discrete Toland-Singer Duality

$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$

$h : \mathbf{Z}^n \to \mathbf{Z}$: M$^\natural$-convex or L$^\natural$-convex (*g*: any fn)

Toland-Singer Duality (Maehara-Murota 14)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^\bullet(p) - g^\bullet(p)\}$$

(Proof) Integral biconjugacy: $h^{\bullet\bullet} = h$.

$$\inf_x\{g(x) - h(x)\} = \inf_x\{g(x) - h^{\bullet\bullet}(x)\}$$
$$= \inf_x\{g(x) - \sup_p\{\langle p, x\rangle - h^\bullet(p)\}\}$$
$$= \inf_x \inf_p\{g(x) - \langle p, x\rangle + h^\bullet(p)\}$$
$$= \inf_p\{h^\bullet(p) - \sup_x\{\langle p, x\rangle - g(x)\}\} = \inf_p\{h^\bullet(p) - g^\bullet(p)\}.$$

Toland-Singer vs Fenchel

$$f^{\bullet}(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\}$$

Toland-Singer Duality (Maehara-Murota 14)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbf{Z}^n} \{h^{\bullet}(p) - g^{\bullet}(p)\}$$

$(g, h) =$ (any, M$^{\natural}$), (any, L$^{\natural}$)

Fenchel Duality (Murota 1996, 1998)

$$\inf_{x \in \mathbf{Z}^n} \{g(x) + h(x)\} = - \inf_{p \in \mathbf{Z}^n} \{h^{\bullet}(p) + g^{\bullet}(-p)\}$$

$(g, h) =$ (M$^{\natural}$, M$^{\natural}$), (L$^{\natural}$, L$^{\natural}$)

containing: Edmonds's matroid intersection
Frank's weight splitting for wtd matroid intersection
Fujishige's Fenchel duality for submod set functions

DC Algorithm (discrete variables)

$$\min_{x\in\mathbb{Z}^n} \{g(x) - h(x)\}$$

Initial point x

$p \leftarrow$ integral subgradient in $\partial h(x) \setminus \partial g(x)$

$h(y) \simeq h(x) + \langle p, y - x \rangle$ (first-order approx)

$x \leftarrow$ solve convex program $\min_{y\in\mathbf{Z}^n} \{g(y) - \langle p, y \rangle\}$

Stop if $g(x) - h(x)$ does not decrease

Optimality Conditions $\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$

x: global opt $\iff$ $\partial_\epsilon g(x) \supseteq \partial_\epsilon h(x) \quad (\forall \epsilon \geq 0)$

[cf. Hiriart-Urruty 1989]

$$\Downarrow$$

$$\boxed{\partial g(x) \supseteq \partial h(x)}$$

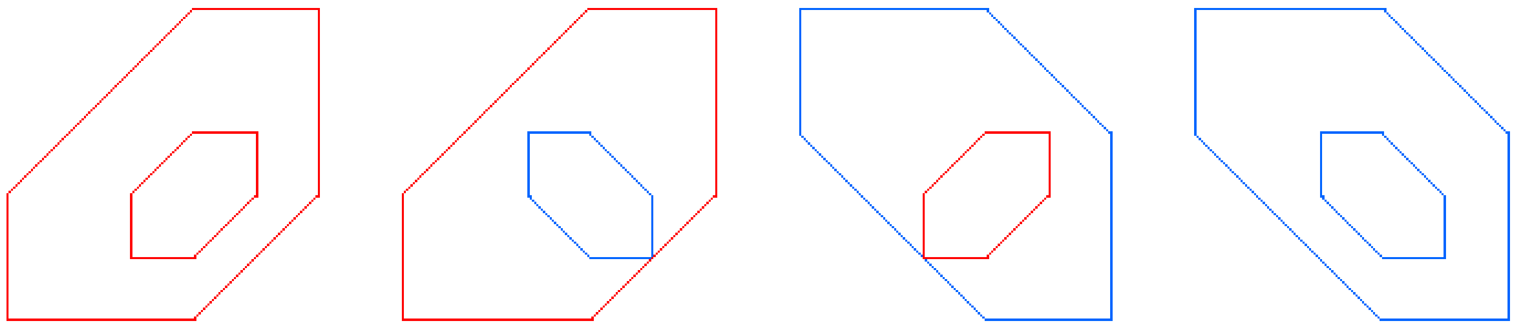
$$\Downarrow$$

x: local opt, i.e., x: minimum in

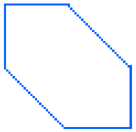
$$U = \bigcup_{p \in \partial g(x)} \partial h^{\bullet}(p)$$

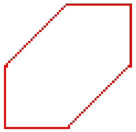
$$\partial_\epsilon f(x) = \{p \mid f(y) - f(x) \geq \langle p, y - x \rangle - \epsilon \quad (\forall y)\}$$

Testing for Local Opt: $\partial g(x) \supseteq \partial h(x)$



M$^\natural$–M$^\natural$	M$^\natural$–L$^\natural$	L$^\natural$–M$^\natural$	L$^\natural$–L$^\natural$
L$^\natural$ $\supseteq$ L$^\natural$	L$^\natural$ $\supseteq$ M$^\natural$	M$^\natural$ $\supseteq$ L$^\natural$	M$^\natural$ $\supseteq$ M$^\natural$
poly-time	**poly-time**	**poly-time**	**co-NP-compl.**
$O(n^2)$ ineqs	$O(n^2)$ M-min	submod-min	(McCormick 96)

 g-polymatroid

 dual of shortest-path

Complexity of Discrete DC Programs

[Maehara-Murota 2015]

$$\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}$$

g, h: "convex" ($\mathrm{dom}\, g \subseteq \mathrm{dom}\, h$)

Convexity: M♮ −M♮, M♮ −L♮, L♮ −M♮, L♮ −L♮

$x \in \mathbb{Z}^n$

$g \setminus h$	M♮	L♮
M♮	NP-hard	NP-hard
L♮	open	NP-hard

pseudo-poly; submod min (ring family)

$x \in \{0, 1\}^n$

$g \setminus h$	M♮	L♮
M♮	NP-hard	NP-hard
L♮	P	NP-hard

Kobayashi 14, submod min

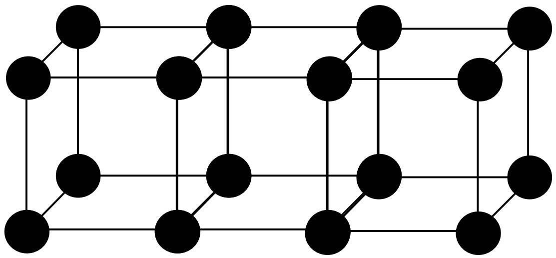
1. DC Programming (continuous)

2. Discrete Convex Analysis

3. DC Programming (M/L-convex)

4. DC Programming (integrally convex)

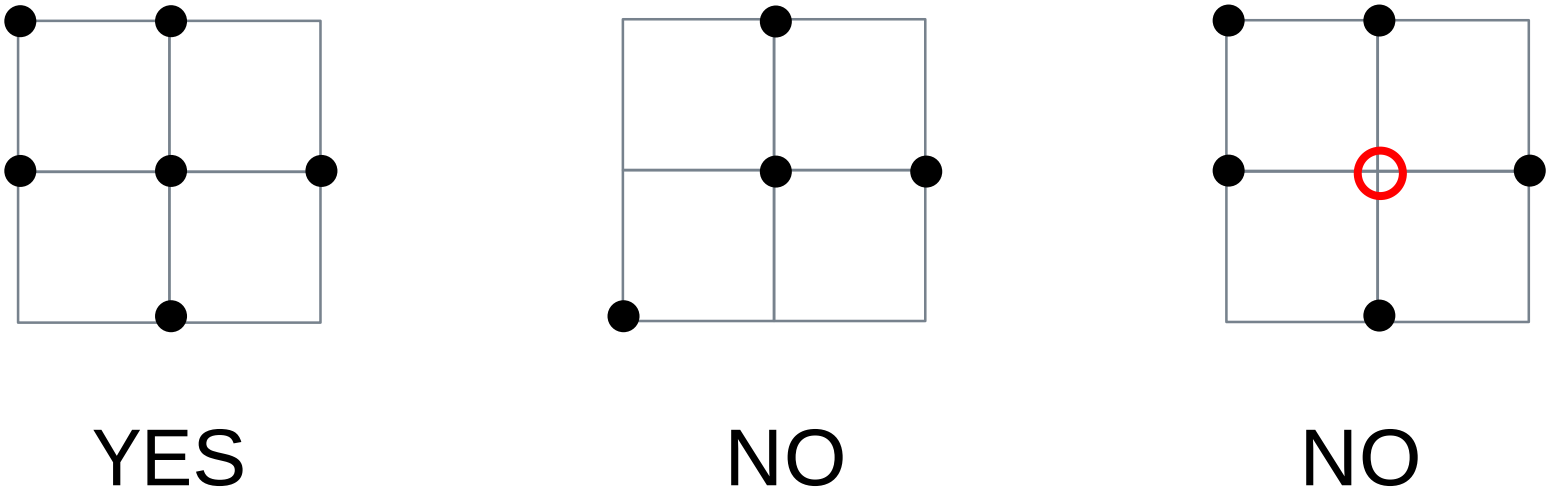
Subclasses of Integrally Convex Functions



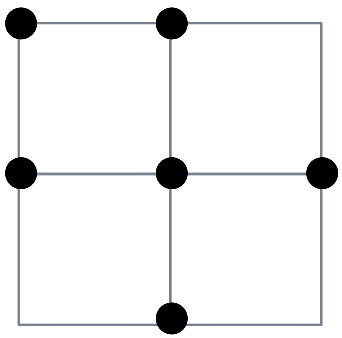
$$\text{Sep-conv} \subset \left\{ \begin{array}{l} \text{M}^{\natural}\text{-conv} \subset \text{M}^{\natural}_{2}\text{-conv} \\ \text{L}^{\natural}\text{-conv} \subset \text{L}^{\natural}_{2}\text{-conv} \end{array} \right\} \subset \text{Int-conv}$$

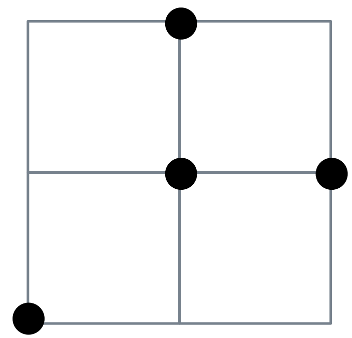
$\text{M}^{\natural}_{2} = \text{M}^{\natural} + \text{M}^{\natural}$ (sum) $\text{L}^{\natural}_{2} = \text{L}^{\natural} \,\square\, \text{L}^{\natural}$ (convolution)

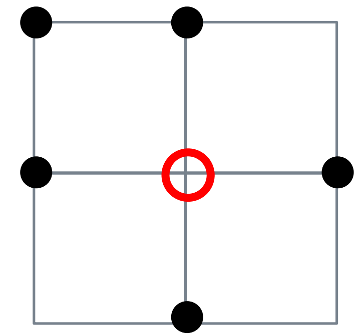
Integrally Convex Set



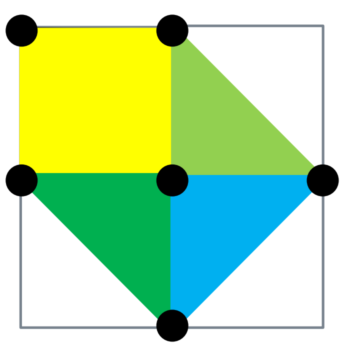
Integrally Convex Set

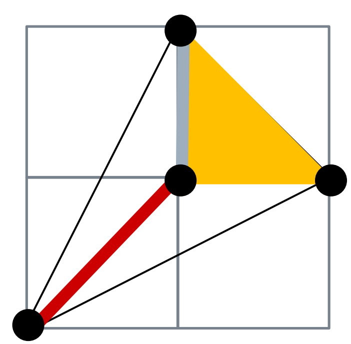






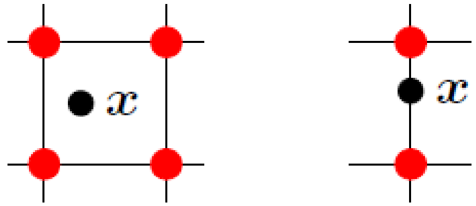
YES NO NO





Integrally Convex Function [Favati-Tardella 90]

$N(x) = \{ z \in \mathbf{Z}^n \mid \|x - z\|_\infty < 1 \}$ integral neighborhood of $x \in \mathbf{R}^n$



Local convex extension of f:

$$\tilde{f}(x) = \sup_{p,\alpha} \{ \langle p, x \rangle + \alpha \mid \langle p, z \rangle + \alpha \leq f(z) \ \ (\forall z \in N(x)) \}$$

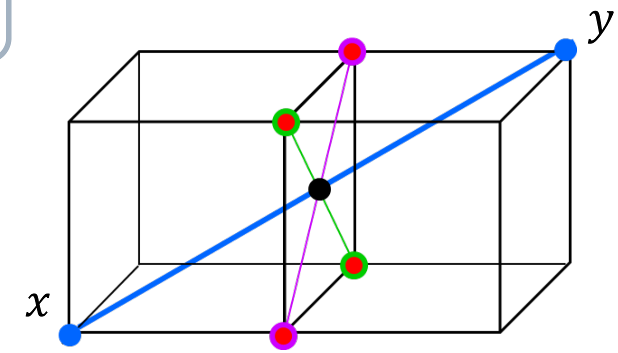
Def. f is integrally convex $\Leftrightarrow$ $\tilde{f}$ is convex

Local Characterization of IC-Functions

local convex extension of f:

$$\tilde{f}(x) = \sup_{p,\alpha}\{\langle p, x\rangle + \alpha \mid \langle p, z\rangle + \alpha \le f(z) \ \ (\forall z \in N(x))\}$$

Def. f is integrally convex ⇔ $\tilde{f}$ is convex



Prop. [Favati-Tardella90] [Moriguchi-Murota-Tamura-Tardella 17]

f is integrally convex (dom f is integrally convex)

⇔ $\forall\, x, y \in \mathrm{dom}\, f, \|x - y\|_\infty = 2$

$$f(x) + f(y) \ge 2\tilde{f}\left(\frac{x+y}{2}\right)$$

Subgradient of Discrete Convex Function

[Murota-Tamura 2018]

Thm f: integrally convex, integer-valued

- Integral subgradient p exists at every point x
- I.e., subdifferential $\partial f(x)$ has an integer point
  (but NOT an integral polyhedron)
- Hence $f^{\bullet\bullet} = f$

[Murota 1998]

Thm f: $\mathrm{M}^{\natural}$-/$\mathrm{L}^{\natural}$-/$\mathrm{M}^{\natural}_2$-/$\mathrm{L}^{\natural}_2$-convex, integer-valued

- Subdifferential $\partial f(x)$ is an integral polyhedron
- Hence integral subgradient p exists at every point x

Subdiff'l of Int-conv: Not Integral Polyhedron

$f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$: integrally-convex

$\not\Rightarrow$ $\partial_{\mathbf{R}} f(x)$: integral polyhedron

Example:

$S = \{(0,0,0), (1,1,0), (0,1,1), (1,0,1)\}$

$f(0,0,0) = 0, \quad f(1,1,0) = f(0,1,1) = f(1,0,1) = 1$

$\partial_{\mathbf{R}} f(0) = \{p \in \mathbf{R}^3 \mid p_1 + p_2 \leq 1, p_2 + p_3 \leq 1, p_1 + p_3 \leq 1\}$

$\partial_{\mathbf{R}} f(0)$ has a vertex at $p = (1/2, 1/2, 1/2)$

Subgradient of Integrally Convex Function

[Murota-Tamura 2018]

Thm f: integrally convex, integer-valued

- Integral subgradient p exists at every point x
- I.e., subdifferential $\partial f(x)$ has an integer point

Proof:

Thm f: integrally convex [Favati-Tardella90]

$x^* \in \arg\min f$ (minimizer)

$\Leftrightarrow \; f(x^*) \leq f(x^* + d) \;\; \forall d \in \{-1, 0, +1\}^n$

+ Fourier-Motzkin elimination

Fourier-Motzkin Elimination (1)

Subdifferential of integrally convex function:

$$\partial_{\mathrm{R}} f(0) = \{p \in \mathrm{R}^n \mid \sum_{j=1}^{n} y_j p_j \leq f(y) \text{ for all } y \in \{-1, 0, +1\}^n\}$$

Inequality system: $Ap \leq b \quad (a_i p \leq b_i \text{ for } i = 1, 2, \ldots)$

$$I_1^+ = \{i \mid a_{i1} = +1\}, \quad I_1^0 = \{i \mid a_{i1} = 0\}, \quad I_1^- = \{i \mid a_{i1} = -1\}$$

Elimination:

$$\begin{array}{rl} +x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots & \leq b_i \\ -x_1 + a_{k2}x_2 + a_{k3}x_3 + \cdots & \leq b_k \\ \hline a'_2 x_2 + a'_3 x_3 + \cdots & \leq b' \end{array}$$

The generated inequalities are redundant by integral convexity

$$\max_{k \in I_1^-} \left\{ \sum_{j=2}^{n} a_{kj} p_j - b_k \right\} \leq p_1 \leq \min_{i \in I_1^+} \left\{ b_i - \sum_{j=2}^{n} a_{ij} p_j \right\}$$

Fourier-Motzkin Elimination (2)

$$\max_{k \in I_1^-} \left\{ \sum_{j=2}^{n} a_{kj} p_j - b_k \right\} \leq p_1 \leq \min_{i \in I_1^+} \left\{ b_i - \sum_{j=2}^{n} a_{ij} p_j \right\},$$

$$\max_{k \in I_2^-} \left\{ \sum_{j=3}^{n} a_{kj} p_j - b_k \right\} \leq p_2 \leq \min_{i \in I_2^+} \left\{ b_i - \sum_{j=3}^{n} a_{ij} p_j \right\},$$

$$\vdots$$

$$\max_{k \in I_{n-1}^-} \{ a_{kn} p_n - b_k \} \leq p_{n-1} \leq \min_{i \in I_{n-1}^+} \{ b_i - a_{in} p_n \},$$

$$\max_{k \in I_n^-} \{ -b_k \} \leq p_n \leq \min_{i \in I_n^+} \{ b_i \}.$$

- b_i and a_{ij} are integers
- $Ap \leq b$ is feasible since $\partial f(0) \neq \emptyset$

Hence we can take integral p

DC Programming for Integrally Convex Function

- **Expressive Power**
  - ➢ Almost all optimization problems
- **Mathematical Properties**
  - ➢ Toland-Singer duality theorem, Optimality conditions
- **Algorithm**
  - ➢ Discrete DC algorithm
  - ➢ Framework works using subgradient
  - ➢ No polynomial complexity

Our Recent Papers on Integral Convexity

- K. Murota and A. Tamura:
  Integrality of subgradients and biconjugates of integrally convex functions (arXiv 1806.00992, 2018)

- S. Moriguchi, K. Murota, A. Tamura, and F. Tardella:
  Scaling, proximity, and optimization of integrally convex functions, Mathematical Programming (2018) (DOI: s10107-018-1234-z)

- S. Moriguchi, K. Murota, A. Tamura, and F. Tardella:
  Discrete midpoint convexity, Mathematics of Operations Research, to appear (arXiv 1708.04579, 2017)

- S. Moriguchi and K. Murota:
  Projection and convolution operations for integrally convex functions, Discrete Applied Mathematics (2018) (DOI: 10.1016/j.dam.2018.08.010)

END