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Discrete Convex Analysis View on Discrete Decreasing Minimization

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Lexico-Optimization on Discrete Sets

Discrete Sets:

- M-convex set (integer pts in integ. base polyhedron)
- Intersection of two M-convex sets
- Network flow
- Submodular flow

Applications: [balancing, fairness, egalitarian]

- Graph orientation
- Resource allocation (in OR, computer science)
- Economics and game theory

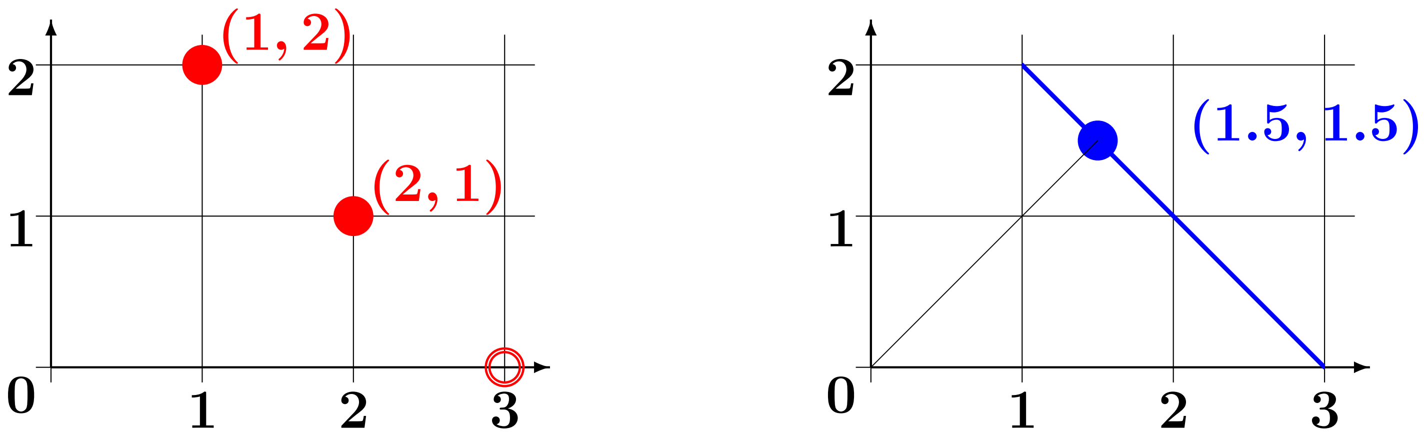
Contents

1. Dec-Minimization via Convex Optimization
2. Optimality Condition via M-optimality Criterion
3. Min-max Formulas via Fenchel-type Duality
4. Structure of Dec-min via M-convex Intersection
5. Discrete vs Continuous

Dec-Min: Discrete ($\mathbb{Z}$) vs Continuous ($\mathbb{R}$)

($\mathbb{Z}$) M-convex set

Base-polyhedron ($\mathbb{R}$)



dec-min

dec-min

= min-norm

(Fujishige, 1980)

1.

Dec-Minimization via Convex Optimization

Decreasing-Minimality: Definition

$x = (2, 5, 5, 1, 4)$ $\qquad$ $x{\downarrow} = (5, 5, 4, 2, 1)$

$y = (1, 5, 5, 5, 1)$ $\qquad$ $y{\downarrow} = (5, 5, 5, 1, 1)$

Def: x is decreasingly smaller than y $\quad$ ($x <_{\rm dec} y$)

$\iff$ $x{\downarrow}$ is lexicographically smaller than $y{\downarrow}$

Def: (Q: set of vectors)

$x \in Q$ is decreasingly minimal (dec-min) in Q

$\iff$ $x \leq_{\rm dec} y$ for all $y \in Q$

Def: inc-max by $x{\uparrow} = (1, 2, 4, 5, 5)$

- inc-max = lexico-optimal (Megiddo, Fujishige)
- dec-min = co-lexico-optimal (Fujishige)

Concept of Majorization

Def: x is majorized by y ($x \prec y$) "more uniform"

$$\sum_{i=1}^{k} x{\downarrow}(i) \leq \sum_{i=1}^{k} y{\downarrow}(i) \ \ (\forall k), \qquad \sum_{i=1}^{n} x{\downarrow}(i) = \sum_{i=1}^{n} y{\downarrow}(i)$$

(sum of k-largest components)

Def: (Q: set of vectors)

$x \in Q$ is least majorized in Q $\iff$ $x \prec y$ for all $y \in Q$

Fact 1: least majorized element may NOT exist

Ex: $Q = \{(2,2,2,0),\ (3,1,1,1)\}$ (M-conv $\cap$ M-conv)
dec-min inc-max $\nexists$ least majorized

Fact 2: least majorized $\Rightarrow$ dec-min & inc-max

Dec-Min by Convex Optimization

Def: $\varphi : \mathbb{Z} \to \mathbb{R}$ **"rapidly increasing"**

$\iff$ $\varphi(k+1)$ is 'much larger' than $\varphi(k)$

$$\varphi(k+1) \geq N\ \varphi(k) > 0 \qquad (N \geq |S|)$$

Prop: Any $D \subseteq \mathbb{Z}^S$, $m \in D$, φ rapid increas. convex

m is **dec-min** $\iff$ m **minimizes** $\sum \varphi(x(s))$ on D

Can use Discrete Convex Analysis for discrete dec-minimization

Def: $\varphi : \mathbb{Z} \to \overline{\mathbb{R}}$ **convex** $\iff$ $\varphi(k-1) + \varphi(k+1) \geq 2\varphi(k)$

2.

Optimality Condition (for Dec-Min) via M-optimality Criterion

Separable convex functions in DCA

$$\text{Min} \quad \Phi(x) = \sum_{s \in S} \varphi_s(x(s)) \quad \text{s.t.} \quad x \in B$$

M-convex set

φ_s: convex $\quad \varphi_s(k-1) + \varphi_s(k+1) \geq 2\varphi_s(k)$

Thm: (Groenevelt, 1985,1991)

$m \in B$ is a minimizer of Φ (global opt)

$$\iff \varphi_s(m(s)+1) + \varphi_t(m(t)-1) \geq \varphi_s(m(s)) + \varphi_t(m(t))$$

if $\quad m + \chi_s - \chi_t \in B$ (local opt)

DCA view:

- A special case of M-optimality criterion (Mu.1996)
- Separ. conv on M-conv set is M-convex function

Optimality Conditions for Dec-Min

As a consequence of Groenevelt's theorem:

Thm: B: M-convex set, $m \in B$,

m is dec-min in B

$\Leftrightarrow m(s) \geq m(t) - 1$ if $m + \chi_s - \chi_t \in B$

$\Leftrightarrow m$ minimizes every symm. separ. convex fn $\cdots (*)$

$\Leftrightarrow m$ minimizes square-sum $\sum x(s)^2$

Fact: $(*) \iff m$ is least majorized

Cor: $\exists$ least majorized element in M-convex set

- Known to experts (Tamir, Ando) ca. 1995

3.

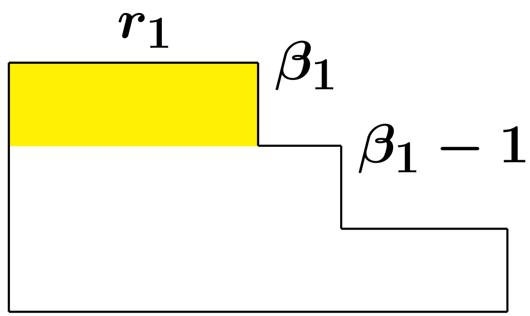
Min-max Formulas (for Dec-Min) via Fenchel-type Duality

Min-Max Formulas for Dec-Min (Frank+Mu.)

- Largest compnt β_1 of dec-min m: (p: supermod.)

$$\beta_1 = \max\{ \lceil p(X)/|X| \rceil : X \neq \emptyset \}$$

- $r_1 = |\{s : m(s) = \beta_1\}|$ for dec-min m:



$$r_1 = \max_X \{ p(X) - (\beta_1 - 1)|X| \}$$

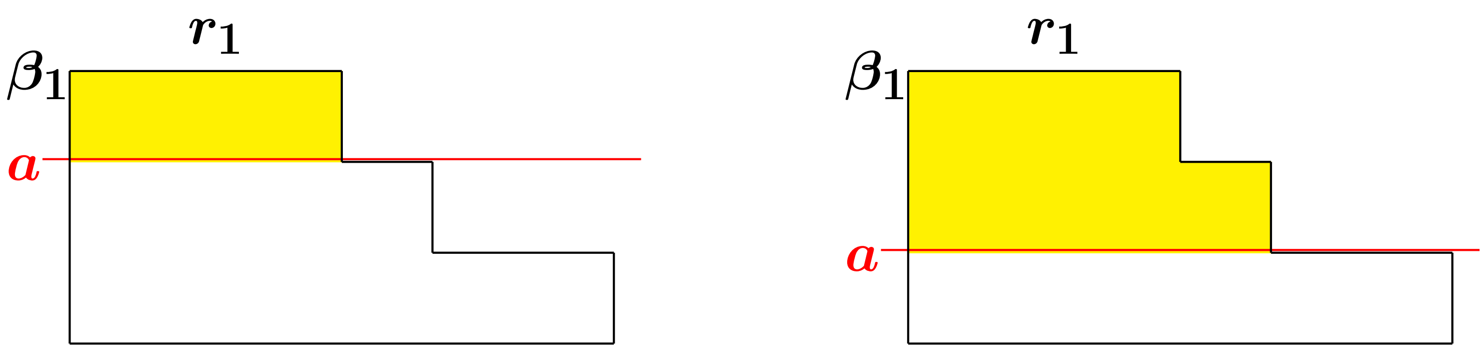
- Square-sum: ($\hat{p}$ = Lovász ext.)

$$\min_{x \in B} \{ \sum_{s \in S} x(s)^2 \} = \max_{\pi \in \mathbb{Z}^S} \{ \hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil \}$$

Min-Max Formulas (cont.)

- **Total a-excess for each $a \in \mathbb{Z}$:**

$$\min\{\sum_{s \in S} (x(s) - a)^+ : x \in B\} = \max_X \{\ p(X) - a|X|\ \}$$



DCA Strategy for Min-Max Formulas

Fenchel-type discrete duality theorem (Mu.1998)

$$\Downarrow$$

Min-max for separable convex fn on an M-convex set

$$\min_{x \in B} \{ \sum_{s} \varphi(x(s)) \} = \max_{\pi \in \mathbb{Z}^S} \{ \hat{p}(\pi) - \sum_{s} \psi(\pi(s)) \}$$

conjugate

$\varphi : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ $\qquad \psi(\ell) = \max\{k\ell - \varphi(k) : k \in \mathbb{Z}\}$

$$\Downarrow$$

Choose φ and calculate ψ to obtain formulas like:

- $\beta_1 = \max\{ \lceil p(X)/|X| \rceil \}$
- $r_1 = \max\{ p(X) - (\beta_1 - 1)|X| \}$
- $\min\{\sum x(s)^2\} = \max\{ \hat{p}(\pi) - \sum \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil \}$
- $\min\{\sum (x(s) - a)^+\} = \max\{ p(X) - a|X| \}$

Proof of Formula for Square-Sum

$$\min_{x\in B}\{\sum_{s}\varphi(x(s))\} = \max_{\pi\in\mathbb{Z}^S}\{\ \hat{p}(\pi) - \sum_{s}\psi(\pi(s))\ \}$$

$$\Downarrow$$

Choose $\varphi(k) = k^2$ and calculate its conjugate:

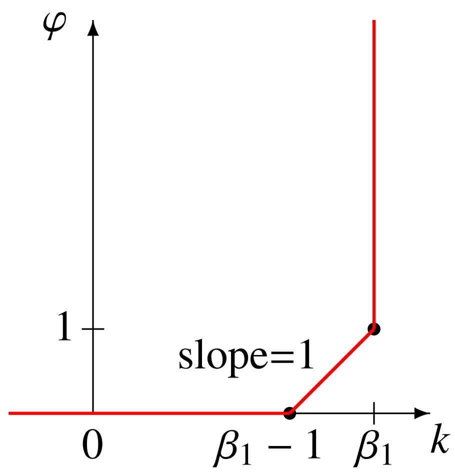
$$\begin{aligned}\psi(\ell) &= \max\{k\ell - k^2 : k \in \mathbb{Z}\} \\ &= \max\{k\ell - k^2 : k \in \{\lfloor \ell/2 \rfloor, \lceil \ell/2 \rceil\}\} \\ &= \left\lfloor \frac{\ell}{2} \right\rfloor \cdot \left\lceil \frac{\ell}{2} \right\rceil\end{aligned}$$

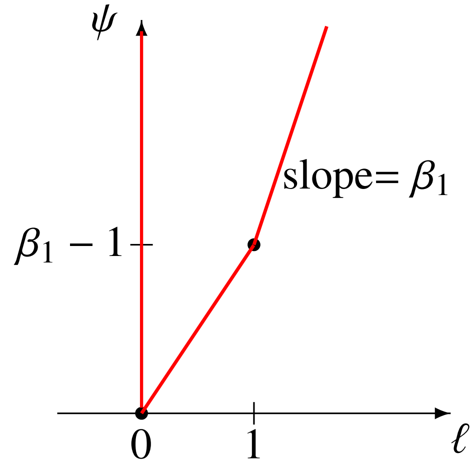
$$\Downarrow$$

$$\min_{x\in B}\{\sum_{s\in S} x(s)^2\} = \max_{\pi\in\mathbb{Z}^S}\{\ \hat{p}(\pi) - \sum_{s\in S}\left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil\ \}$$

DCA offers a method of discovery (and proof)

Proof of Formula for r_1





$$\varphi(k) = \begin{cases} 0 & (k \leq \beta_1 - 1) \\ 1 & (k = \beta_1) \\ +\infty & (k \geq \beta_1 + 1) \end{cases}$$

$$\psi(\ell) = \begin{cases} +\infty & (\ell \leq -1) \\ 0 & (\ell = 0) \\ \beta_1 \ell - 1 & (\ell \geq 1) \end{cases}$$

$r_1 := |\{s : m(s) = \beta_1\}|$ for dec-min m

$$||$$

$$\min_{x \in B}\{ \sum \varphi(x(s)) \} = \max_{\pi \in \mathbb{Z}^S}\{ \hat{p}(\pi) - \sum \psi(\pi(s)) \}$$

L$^\natural$-concave

$$|| \qquad \leftarrow \pi \in \{0,1\}^S$$

$$\max\{p(X) - (\beta_1 - 1)|X|\}$$

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation $f(x) \geq \boxed{\text{Lin}} \geq h(x)$	**Fenchel duality** (Fujishige 84) **matroid intersect.** (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\} = \sup\{h^\circ - f^\bullet\}$	$\Rightarrow$ **discrete separ. for submod** (Frank 82) $\Rightarrow$ **valuated matroid intersect.** (Mu. 96)
$\Updownarrow$	$\Downarrow$
L-separation $f^\bullet(p) \geq \boxed{\text{Lin}} \geq h^\circ(p)$	**weighted matroid intersect.** (Lawler 75, Iri-Tomizawa 76, Edmonds 79, Frank 81)

4.
Structure of Dec-min via M-convex Intersection

Matroidal Structure of Dec-min Elements

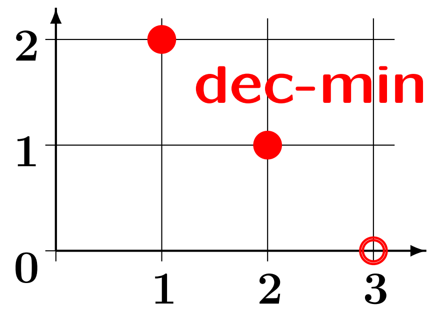
dm(B): set of the dec-min elements of B

Thm: (Frank+Mu.)

dm(B) is a matroidal M-convex set:

$\exists$ matroid $\hat{M}$, $\exists$ integer vector $\hat{\Delta}$:

$$\text{dm}(B) = \{\chi_L + \hat{\Delta} : L \text{ is a basis of } \hat{M}\}$$

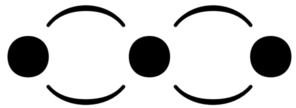


Example: $B = \{m_1, m_2, m_3, m_4, m_5\}$

$m_1 = (2,1,1,0)$, $m_2 = (2,1,0,1)$

$m_3 = (1,2,1,0)$, $m_4 = (1,2,0,1)$ dec-min ($\mathbb{Z}$)

$m_5 = (2,2,0,0)$ NOT dec-min ($\mathbb{Z}$)

$\hat{M}$: graphic matroid , $\hat{\Delta} = (1,1,0,0)$

M-convex Intersection Theorem in DCA

Min $f(x) + g(x)$ $\qquad$ f, g: M$^\natural$-convex, $\mathbb{Z}$-valued

Thm (M-convex intersection): **(Mu. 1996)**

$\exists \hat{\pi} \in \mathbb{Z}^n$ **(opt certificate, subgradient)**

$$\mathrm{argmin}\,(f(x) + g(x))$$

$$= \mathrm{argmin}\,(f(x) - \langle \hat{\pi}, x \rangle) \ \cap \ \mathrm{argmin}\,(g(x) + \langle \hat{\pi}, x \rangle)$$

extension of weight-splitting

$$\Downarrow$$

Choose $f(x) = \sum x(s)^2$, $\quad g(x) = \delta(x)$ **(indicator of** B**)**

$$\Downarrow$$

$$\mathbf{dm}(B) = \arg\min_{x \in B} \{\sum x(s)^2\} = \mathrm{argmin}\,(f(x) + g(x))$$

$$= \mathrm{argmin}\,(f(x) - \langle \hat{\pi}, x \rangle) \cap \mathrm{argmin}\,(g(x) + \langle \hat{\pi}, x \rangle)$$

$$\Downarrow$$

- For $f(x) = \sum x(s)^2$

$$\mathrm{argmin}\,(f(x) - \langle \hat{\pi}, x \rangle) = \text{Box}\{\lfloor \hat{\pi}(s)/2 \rfloor, \lceil \hat{\pi}(s)/2 \rceil\}$$

- For $g(x) = \delta(x)$ (indicator of B)

$$\mathrm{argmin}\,(g(x) + \langle \hat{\pi}, x \rangle) = \{\hat{\pi}\text{-min elements of } B\}$$

(M-convex set)

$$\Downarrow$$

dm(B) = (unit-sized box) $\cap$ (M-convex set)

= matroidal M-convex set

Q.E.D.

Note: M-convex intersection $\Longleftrightarrow$ Fenchel duality

Canonical Partition and Near-Uniformity

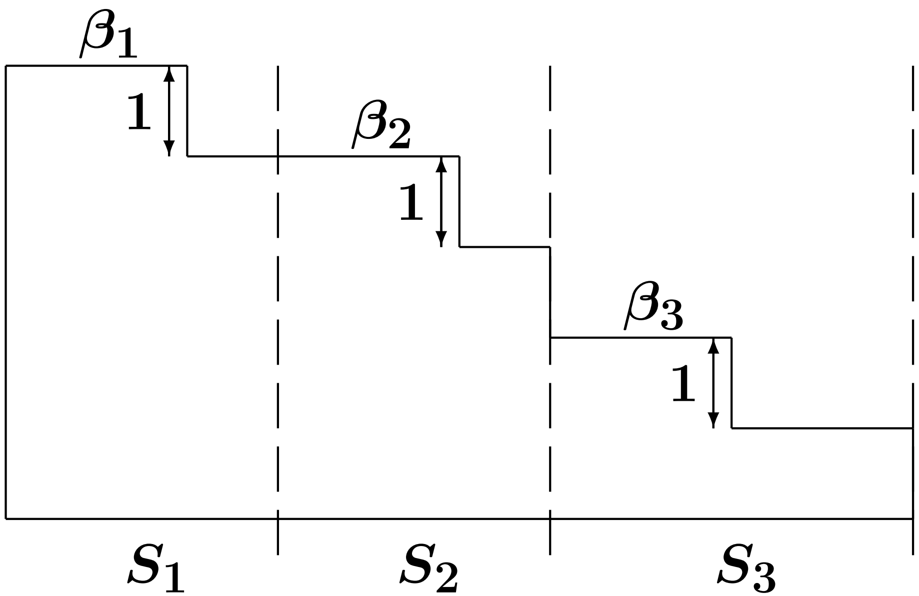
canonical partition $\{S_1, S_2, \ldots, S_q\}$

essential values $\beta_1 > \beta_2 > \cdots > \beta_q$

Thm: $m \in B$ is dec-min (Frank+Mu.)

$\Longleftrightarrow$ • $\beta_i - 1 \leq m(s) \leq \beta_i$ for $s \in S_i$ (near-uniform)

• $C_i := S_1 \cup \cdots \cup S_i$ is tight, i.e., $m(C_i) = p(C_i)$



Canonical Partition: Alternative Construction

- Maximizers of $p(X) - \beta|X|$ form a lattice for $\beta \in \mathbb{Z}$

  Let $L(\beta)$ be the smallest maximizer

- $\exists$ finitely many $\beta \in \mathbb{Z}$ with $L(\beta) \neq L(\beta - 1)$

  $\beta_1 > \beta_2 > \cdots > \beta_q$: essential values

$$\emptyset = L(\beta_1) \subset L(\beta_1 - 1) = \cdots = L(\beta_2) \subset L(\beta_2 - 1)$$
$$= \cdots = L(\beta_q) \subset L(\beta_q - 1) = S$$

- Define $C_j := L(\beta_j - 1) \qquad (j = 1, 2, \ldots, q)$

  $C_1 \subset C_2 \subset \cdots \subset C_q$: canonical chain

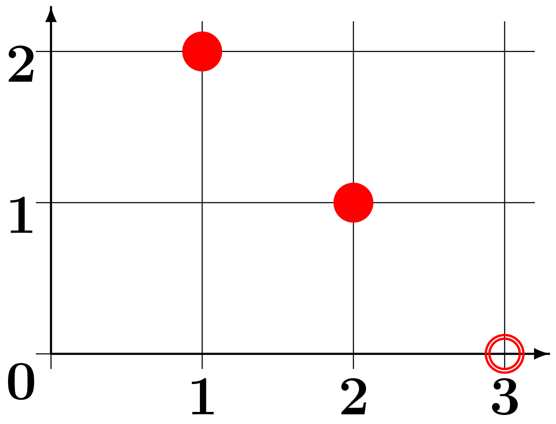
- Define $S_j = C_j - C_{j-1} \qquad (j = 1, 2, \ldots, q), \qquad C_0 = \emptyset$

  $\{S_1, S_2, \ldots, S_q\}$: canonical partition

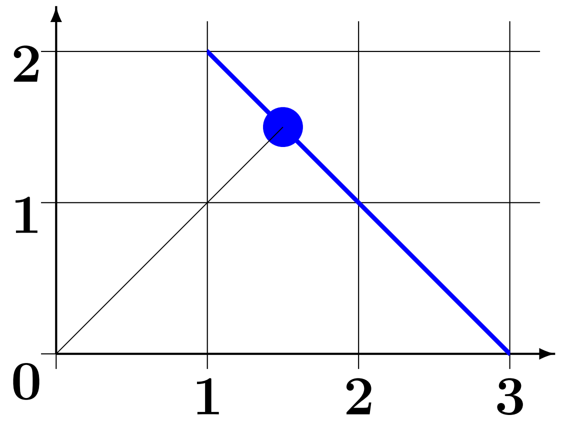
5.
Discrete vs Continuous

Structure of Dec-Min Elements

($\mathbb{Z}$) M-convex set



($\mathbb{R}$) Base-polyhedron



Matroidal M-convex set

(Frank+Mu.)

Unique = min-norm point

(Fujishige)

Canonical Partition vs Principal Partition

Same construction with parameter

Integer $\beta \in \mathbb{Z}$ vs Real $\lambda \in \mathbb{R}$

$\beta_1 > \beta_2 > \cdots > \beta_q$ (essential values) $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ (critical values)

Principal Partition: Japanese tradition, 1968–1980's

Kishi-Kajitani, Ozawa, Iri, Tomizawa, Fujishige, Nakamura/ Bruno-Weinberg, Narayanan (Survey by Fujishige, 2009)

Thm: (Frank+Mu.)

- β is essential value $\iff$ $\exists\lambda$ with $\beta \geq \lambda > \beta - 1$
- Essential values β_j are distinct members of
  $\lceil\lambda_1\rceil \geq \lceil\lambda_2\rceil \geq \cdots \geq \lceil\lambda_r\rceil$ (round-up critical values)
- Canonical parttn is aggregation of princical parttn
- Canonical chain is a subchain of principal chain

Proximity Theorem for Dec-Min

Thm: (Frank+Mu.)

$m_{\mathbb{R}}$: minimum norm point of $B_{\mathbb{R}}$

$\boxed{\forall}$ dec-min element $m_{\mathbb{Z}}$ of $B_{\mathbb{Z}}$ satisfies

$$\lfloor m_{\mathbb{R}} \rfloor \leq m_{\mathbb{Z}} \leq \lceil m_{\mathbb{R}} \rceil$$

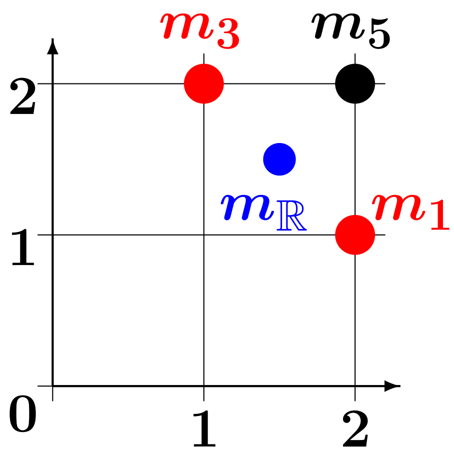
$\longrightarrow$ Continuous relaxation algorithm

using strong-polynomial algo for min-norm pt $m_{\mathbb{R}}$

Example: Proximity Theorem for Dec-Min

$m_{\mathbb{R}}$ minimum norm point of $B_{\mathbb{R}}$

$m \in B_{\mathbb{Z}}$ is dec-min $\left\{ \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \right\}$ $\lfloor m_{\mathbb{R}} \rfloor \leq m \leq \lceil m_{\mathbb{R}} \rceil$



Example: $B_{\mathbb{Z}} = \{m_1, m_2, m_3, m_4, m_5\}$

- $m_1 = (2,1,1,0)$, $m_2 = (2,1,0,1)$
  $m_3 = (1,2,1,0)$, $m_4 = (1,2,0,1)$ **dec-min** ($\mathbb{Z}$)
  $m_5 = (2,2,0,0)$ NOT dec-min ($\mathbb{Z}$)
- $m_{\mathbb{R}} = (3/2, 3/2, 1/2, 1/2)$ min-norm point ($\mathbb{R}$)
- $m_5 = (2,2,0,0)$ satisfies
  $(1,1,0,0) = \lfloor m_{\mathbb{R}} \rfloor \leq m_5 \leq \lceil m_{\mathbb{R}} \rceil = (2,2,1,1)$

Min-Max Formulas: $(\mathbb{Z})$ vs $(\mathbb{R})$

- Largest compnt of dec-min m:

$$(\mathbb{Z})\ \beta_1 = \max_{X \neq \emptyset}\{\ \lceil\ p(X)/|X|\ \rceil\} \qquad (\mathbb{R})\ \lambda_1 = \max_{X \neq \emptyset}\{\ p(X)/|X|\ \}$$

- Total a-excess: $(\mathbb{Z})\ a \in \mathbb{Z}$ $(\mathbb{R})\ a \in \mathbb{R}$

$$\min\{\sum_{s \in S}(x(s) - a)^+ : x \in B\} = \max_{X}\{\ p(X) - a|X|\ \}$$

($B_{\mathbb{Z}}$ or $B_{\mathbb{R}}$)

- Square-sum:

$$(\mathbb{Z})\ \min_{x \in B_{\mathbb{Z}}}\{\sum_{s \in S} x(s)^2\} = \max_{\pi \in \mathbb{Z}^S}\{\ \hat{p}(\pi) - \sum_{s \in S}\left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil\ \}$$

$$(\mathbb{R})\ \min_{x \in B_{\mathbb{R}}}\{\sum_{s \in S} x(s)^2\} = \max_{\pi \in \mathbb{R}^S}\{\ \hat{p}(\pi) - \sum_{s \in S}\left(\frac{\pi(s)}{2}\right)^2\ \}$$

Concluding Remark

Discrete convexity approach works for

- M-convex set (Part I)
- Intersection of two M-convex sets (Part IV)
- Network flow (Part III)
- Submodular flow (Part IV)

- Their weighted versions (Part V)

  (dec-min/lexico-opt with respect to weight vectors)

E N D

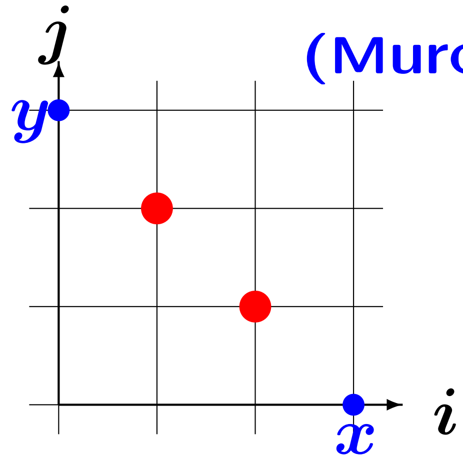
Supplement

M-convex Function (M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector

(Murota 96)

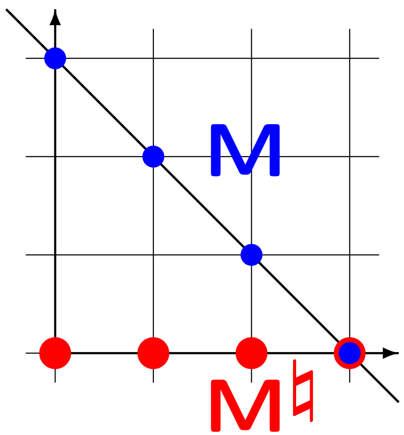


Def: f is M-convex

$\Longleftrightarrow \quad \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom}\, f \subseteq$ const-sum hyperplane



Fenchel-type Duality in DCA

f: M$^\natural$-convex, h: M$^\natural$-concave $(\mathbb{Z}^n \to \mathbb{Z})$

Legendre–Fenchel transform

$$f^\bullet(p) = \max\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$

$$h^\circ(p) = \min\{\langle p, x\rangle - h(x) \mid x \in \mathbb{Z}^n\}$$

Fenchel-type duality thm (Murota 96, 98)

$$\min_{x \in \mathbb{Z}^n}\{f(x) - h(x)\} = \max_{p \in \mathbb{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

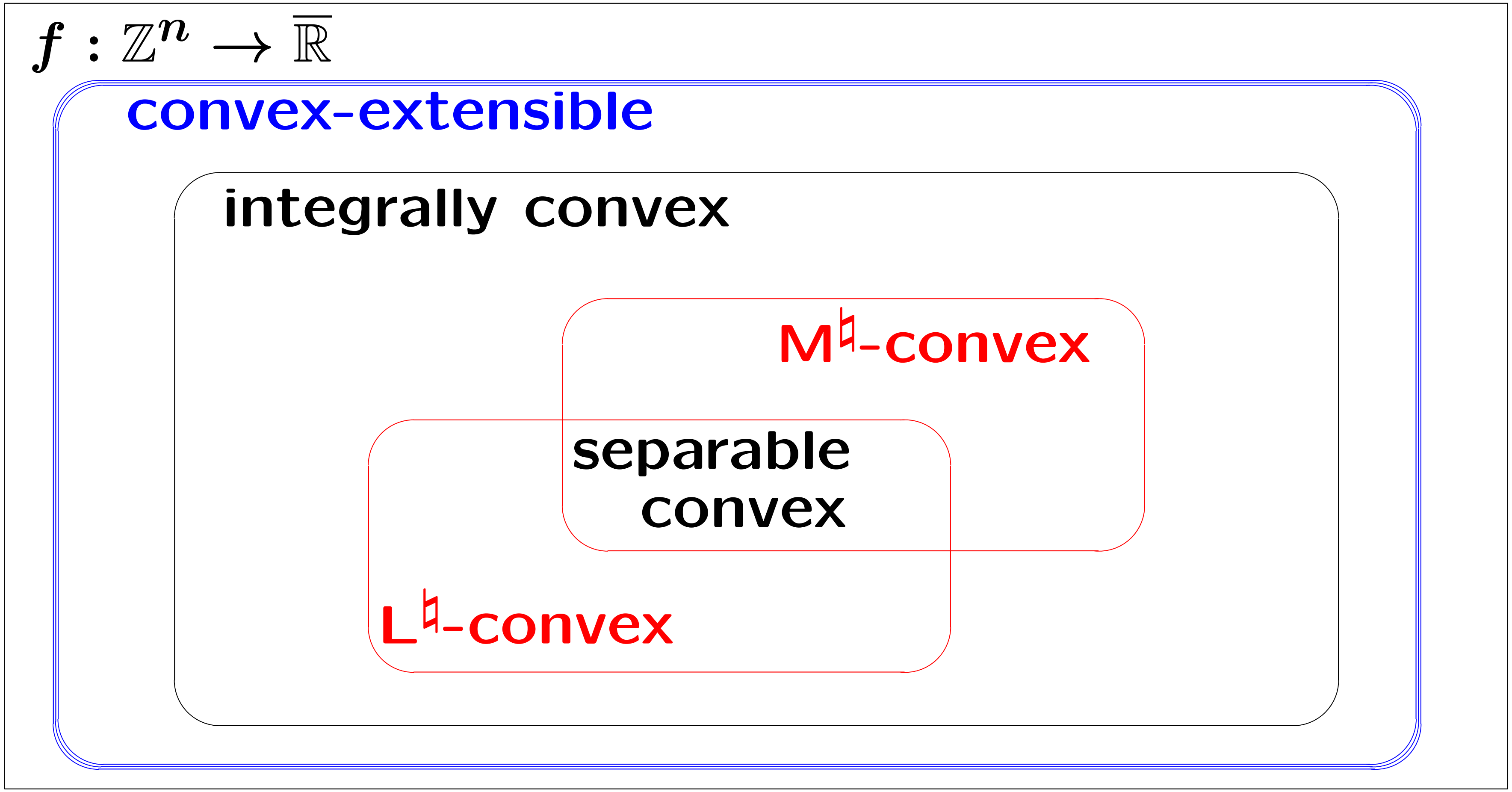
Referencing Relations between Papers

	Vei	Meg	Fuj	Gro	F-G	I-K	D-R	Fuj	Hoc	Tam
Veinott 1971	·	–	–	–	–	–	–	–	–	–
Megiddo 1974	–	·	–	–	–	–	–	–	–	–
Fujishige 1980	–	R	·	–	–	–	–	–	–	–
Groenevelt 1985/91	–	R	R	·	R	–	–	–	–	–
Feder.–Groen.1986	–	R	R	–	·	–	–	–	–	–
Ibaraki–Katoh 1988	–	R	R	R	R	·	–	–	–	–
Dutta–Ray 1989	–	–	–	–	–	–	·	–	–	–
Fujishige 1991	–	R	R	R	–	R	$\mathrm{R}^{\mathrm{2nd}}$	·	–	–
Hochbaum 1994	–	–	–	R	R	R	–	–	·	–
Tamir 1995	R	R	R	R	–	–	R	R	–	·

Paper at the left refers to papers marked R in the same row

$\mathrm{R}^{\mathrm{2nd}}$ means that reference is made in the 2nd edition (2005) only

Classes of Discrete Convex Functions



M$^\natural$ $\cap$ L$^\natural$ $=$ separable