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Discrete Convex Analysis*

Kazuo Murota

1 Introduction

Discrete convex analysis [18, 40, 43, 47] aims to establish a general theoretical framework for solvable discrete optimization problems by means of a combination of the ideas in continuous optimization and combinatorial optimization. The framework of convex analysis is adapted to discrete settings and the mathematical results in matroid/submodular function theory are generalized. Viewed from the continuous side, it is a theory of convex functions $f : \mathbf{R}^n \to \mathbf{R}$ that have additional combinatorial properties. Viewed from the discrete side, it is a theory of discrete functions $f : \mathbf{Z}^n \to \mathbf{R}$ or $f : \mathbf{Z}^n \to \mathbf{Z}$ that enjoy certain nice properties comparable to convexity. Symbolically,

$$\text{Discrete Convex Analysis} \;=\; \text{Convex Analysis} \;+\; \text{Matroid Theory}.$$

The theory extends the direction set forth by J. Edmonds [11], A. Frank [16], S. Fujishige [17], and L. Lovász [34]. The reader is referred to [59] for convex analysis, [8, 32, 60] for combinatorial optimization, [57, 58, 71] for matroid theory, and [18, 70] for submodular function theory.

Two convexity concepts, called L-convexity and M-convexity, play primary roles. L-convex functions and M-convex functions are conjugate to each other through the (continuous or discrete) Legendre–Fenchel transformation. L-convex functions and M-convex functions generalize, respectively, the concepts of submodular set functions and base polyhedra. It is noted that “L” stands for “Lattice” and “M” for “Matroid.”

The set of all real numbers is denoted by $\mathbf{R}$, and $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ and $\underline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\}$. The set of all integers is denoted by $\mathbf{Z}$, and $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{+\infty\}$ and $\underline{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty\}$. Let $V = \{1, 2, \ldots, n\}$ for a positive integer n. The characteristic vector of $X \subseteq V$ is denoted by $\chi_X \in \{0, 1\}^n$. For $i \in V$, we write χ_i for $\chi_{\{i\}}$, which is the ith unit vector, and $\chi_0 = \mathbf{0}$ (zero vector).

2 Concepts of Discrete Convex Functions

The concepts of L-convex and M-convex functions can be obtained through discretization of two different characterizations of convex functions.

2.1 Ordinary Convex Functions

We start by recalling the definition of ordinary convex functions. A function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to be *convex* if

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) \tag{1}$$

for all $x, y \in \mathbf{R}^n$ and for all λ with $0 \leq \lambda \leq 1$, where it is understood that the inequality is satisfied if $f(x)$ or $f(y)$ is equal to $+\infty$. A function $h : \mathbf{R}^n \to \underline{\mathbf{R}}$ is said to be *concave* if $-h$ is convex.

A set $S \subseteq \mathbf{R}^n$ is called *convex* if, for any $x, y \in S$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda) y \in S$. The *indicator function* of a set S is a function $\delta_S : \mathbf{R}^n \to \{0, +\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases} \tag{2}$$

*This is a reproduction of Sections 1 to 6, with minor revisions, of [K. Murota: Recent developments in discrete convex analysis, *Research Trends in Combinatorial Optimization, Bonn 2008* (W. Cook, L. Lovász and J. Vygen, eds.), Springer-Verlag, Berlin, 2009, Chapter 11, 219–260], prepared as a supplementary reading material for Summer School on Combinatorial Optimization (September 21–25, 2015) at Hausdorff Institute of Mathematics.

Then S is a convex set if and only if δ_S is a convex function.

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty, +\infty\}$ in general, the set

$$\mathrm{dom}_{\mathbf{R}} f = \{x \in \mathbf{R}^n \mid f(x) \in \mathbf{R}\}$$

is called the *effective domain* of f. A point $x \in \mathbf{R}^n$ is said to be a *global minimum* of f if the inequality $f(x) \le f(y)$ holds for every $y \in \mathbf{R}^n$. Point x is a *local minimum* if this inequality holds for every y in some neighborhood of x. The set of global minima (minimizers) is denoted as

$$\mathrm{argmin}_{\mathbf{R}} f = \{x \in \mathbf{R}^n \mid f(x) \le f(y)\ (\forall y \in \mathbf{R}^n)\}.$$

Convex functions are tractable in optimization (or minimization) problems and this is mainly because of the following properties.

1. Local optimality (or minimality) guarantees global optimality.
2. Duality theorems such as min-max relation and separation hold.

Duality is a central issue in convex analysis, and is discussed in Section 5.

A *separable convex function* is a function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ that can be represented as

$$f(x) = \sum_{i=1}^{n} \varphi_i(x_i), \tag{3}$$

where $x = (x_i \mid i = 1, \ldots, n)$ and $\varphi_i : \mathbf{R} \to \overline{\mathbf{R}}$ $(i = 1, \ldots, n)$ are univariate convex functions.

2.2 Discrete Convex Functions

We now consider how convexity concept can (or should) be defined for functions in discrete variables. It would be natural to expect the following properties of any function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ that is qualified as a "discrete convex function."

1. Function f is extensible to a convex function on $\mathbf{R}^n$.
2. Local optimality (or minimality) guarantees global optimality.
3. Duality theorems such as min-max relation and separation hold.

Recall that $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be *convex-extensible* if there exists a convex function $\overline{f} : \mathbf{R}^n \to \overline{\mathbf{R}}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$. It is widely understood that convex extensibility alone does not yield a fruitful theoretical framework, which fact motivates us to introduce L-convex and M-convex functions. In this section we focus on convex extensibility and local optimality while deferring duality issues to Section 5. The effective domain and the set of minimizers are denoted respectively as

$$\begin{aligned} \mathrm{dom}_{\mathbf{Z}} f &= \{x \in \mathbf{Z}^n \mid f(x) \in \mathbf{R}\}, \\ \mathrm{argmin}_{\mathbf{Z}} f &= \{x \in \mathbf{Z}^n \mid f(x) \le f(y)\ (\forall y \in \mathbf{Z}^n)\}. \end{aligned}$$

2.2.1 Univariate and separable convex functions

The univariate case ($n = 1$) is simple and straightforward. We may regard a function $f : \mathbf{Z} \to \overline{\mathbf{R}}$ as a discrete convex function if

$$f(x-1) + f(x+1) \ge 2f(x) \qquad (\forall x \in \mathbf{Z}). \tag{4}$$

This is justified by the following facts.

Theorem 1. *A function $f : \mathbf{Z} \to \overline{\mathbf{R}}$ is convex-extensible if and only if it satisfies* (4).

Theorem 2. *For a function $f : \mathbf{Z} \to \overline{\mathbf{R}}$ satisfying* (4)*, a point $x \in \mathrm{dom}_{\mathbf{Z}} f$ is a global minimum if and only if it is a local minimum in the sense that*

$$f(x) \le \min\{f(x-1), f(x+1)\}.$$

Theorems 1 and 2 above can be extended in obvious ways to a *separable (discrete) convex function* $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$, which is, by definition, representable in the form of (3) with univariate functions $\varphi_i : \mathbf{Z} \to \overline{\mathbf{R}}$ having property (4).

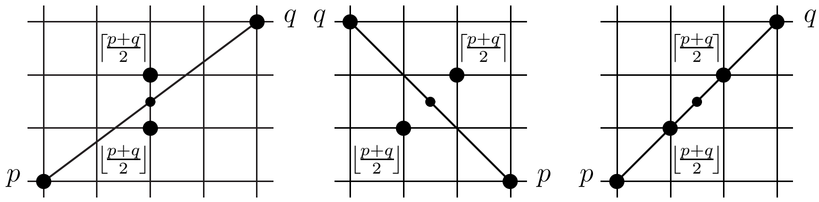

Figure 1: Discrete midpoint convexity

2.2.2 L-convex functions

We explain the concept of L-convex functions [40] by featuring an equivalent variant thereof, called L$^\natural$-convex functions [19] ("L$^\natural$" should be read "el natural").

We first observe that a convex function g on $\mathbf{R}^n$ satisfies

$$g(p) + g(q) \geq g\left(\frac{p+q}{2}\right) + g\left(\frac{p+q}{2}\right) \qquad (p, q \in \mathbf{R}^n), \tag{5}$$

which is a special case of (1) with $\lambda = 1/2$. This property, called *midpoint convexity*, is known to be equivalent to convexity if g is a continuous function.

For a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in discrete variables the above inequality does not always make sense, since the midpoint $\frac{p+q}{2}$ of two integer vectors p and q may not be integral. Instead we simulate (5) by

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \qquad (p, q \in \mathbf{Z}^n), \tag{6}$$

where, for $z \in \mathbf{R}$ in general, $\lceil z \rceil$ denotes the smallest integer not smaller than z (rounding-up to the nearest integer) and $\lfloor z \rfloor$ the largest integer not larger than z (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications, as illustrated in Fig. 1 in the case of $n = 2$. We refer to (6) as *discrete midpoint convexity* [12].

We say that a function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is *L$^\natural$-convex* if it satisfies discrete midpoint convexity (6). In the case of $n = 1$, L$^\natural$-convexity is equivalent to the condition (4). Examples of L$^\natural$-convex functions are given in Section 4.1.

With this definition we can obtain the following desired statements in parallel with Theorems 1 and 2.

Theorem 3. *An L$^\natural$-convex function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is convex-extensible.*

Theorem 4. *For an L$^\natural$-convex function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$, a point $p \in \mathrm{dom}_{\mathbf{Z}} g$ is a global minimum if and only if it is a local minimum in the sense that*

$$g(p) \leq \min\{g(p - q), g(p + q)\} \qquad (\forall q \in \{0, 1\}^n). \tag{7}$$

Although Theorem 4 affords a local criterion for global optimality of a point p, a straightforward verification of (7) requires $\mathrm{O}(2^n)$ function evaluations. The verification can be done in polynomial time as follows. We consider set functions ρ_p^+ and ρ_p^- defined by $\rho_p^\pm(Y) = g(p \pm \chi_Y) - g(p)$ for $Y \subseteq V$, both of which are submodular. Since (7) is equivalent to saying that both ρ_p^+ and ρ_p^- achieve the minimum at $Y = \emptyset$, this condition can be verified in polynomial time by submodular function minimization algorithms [24, 35].

L$^\natural$-convexity is closely related with submodularity. For two vectors p and q, the vectors of componentwise maxima and minima are denoted respectively by $p \vee q$ and $p \wedge q$, that is,

$$(p \vee q)_i = \max(p_i, q_i), \qquad (p \wedge q)_i = \min(p_i, q_i).$$

A function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is called *submodular* if

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbf{Z}^n), \tag{8}$$

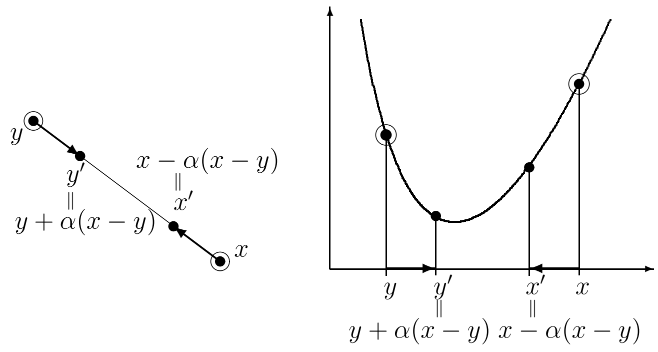

Figure 2: Equidistance convexity

and *translation submodular* if

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \quad (\alpha \in \mathbf{Z}_+,\ p, q \in \mathbf{Z}^n), \tag{9}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{Z}_+$ denotes the set of nonnegative integers. The latter property characterizes L$^\natural$-convexity, as follows.

Theorem 5. *For a function* $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$*, translation submodularity* (9) *is equivalent to discrete midpoint convexity* (6)*.*

An *L-convex function* is defined as an L$^\natural$-convex function g that satisfies

$$g(p + \mathbf{1}) = g(p) + r \tag{10}$$

for some $r \in \mathbf{R}$ (which is independent of p). It is known that g is L-convex if and only if it satisfies (8) and (10); in fact this is the original definition of L-convexity. L-convex functions and L$^\natural$-convex functions are essentially the same, in that L$^\natural$-convex functions in n variables can be identified, up to the constant r in (10), with L-convex functions in $n + 1$ variables.

2.2.3 M-convex functions

Just as L-convexity is defined through discretization of midpoint convexity, another kind of discrete convexity, called M-convexity [39, 40], can be defined through discretization of another convexity property. We feature an equivalent variant of M-convexity, called M$^\natural$-convexity [49] ("M$^\natural$" should be read "em natural").

We first observe that a convex function f on $\mathbf{R}^n$ satisfies the inequality

$$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y)) \tag{11}$$

for every $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq 1$. This inequality follows from (1) for $\lambda = \alpha$ and $\lambda = 1 - \alpha$, whereas it implies (1) if f is a continuous function. The inequality (11) says that the sum of the function values evaluated at two points, x and y, does not increase if the two points approach each other by the same distance on the line segment connecting them (see Fig. 2). We refer to this property as *equidistance convexity*.

For a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in discrete variables we simulate equidistance convexity (11) by moving a pair of points (x, y) to another pair (x', y') along the coordinate axes rather than on the connecting line segment. To be more specific, we consider two kinds of possibilities

$$(x', y') = (x - \chi_i, y + \chi_i) \quad \text{or} \quad (x', y') = (x - \chi_i + \chi_j, y + \chi_i - \chi_j) \tag{12}$$

with indices i and j such that $x_i > y_i$ and $x_j < y_j$; see Fig. 3. For a vector $z \in \mathbf{R}^n$ in general, define the *positive* and *negative supports* of z as

$$\mathrm{supp}^+(z) = \{i \mid z_i > 0\}, \qquad \mathrm{supp}^-(z) = \{j \mid z_j < 0\}.$$

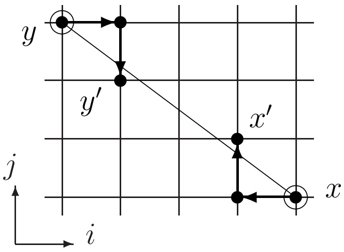

Figure 3: Nearer pair in the definition of M$^\natural$-convex functions

Then the expression (12) can be rewritten compactly as $(x', y') = (x - \chi_i + \chi_j, y + \chi_i - \chi_j)$ with $i \in \mathrm{supp}^+(x - y)$ and $j \in \mathrm{supp}^-(x - y) \cup \{0\}$, where χ_0 is defined to be the zero vector.

As a discrete analogue of equidistance convexity (11) we consider the following condition: For any $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ and any $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j), \tag{13}$$

which is referred to as the *exchange property*. A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ having this exchange property is called *M$^\natural$-convex*. In the case of $n = 1$, M$^\natural$-convexity is equivalent to the condition (4). Examples of M$^\natural$-convex functions are given in Section 4.2.

With this definition we can obtain the following desired statements comparable to Theorems 1 and 2.

Theorem 6. *An M$^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is convex-extensible.*

Theorem 7. *For an M$^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$, a point $x \in \mathrm{dom}_{\mathbf{Z}} f$ is a global minimum if and only if it is a local minimum in the sense that*

$$f(x) \leq f(x - \chi_i + \chi_j) \qquad (\forall i, j \in \{0, 1, \ldots, n\}).$$

An *M-convex function* is defined as an M$^\natural$-convex function f that satisfies (13) with $j \in \mathrm{supp}^-(x - y)$. This is equivalent to saying that f is an M-convex function if and only if it is M$^\natural$-convex and $\mathrm{dom}_{\mathbf{Z}} f \subseteq \{x \in \mathbf{Z}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbf{Z}$. M-convex functions and M$^\natural$-convex functions are essentially the same, in that M$^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n + 1$ variables.

2.2.4 Classes of discrete convex functions

We have thus defined L$^\natural$-convex functions and M$^\natural$-convex functions by discretization of midpoint convexity and equidistance convexity, respectively. The definitions are summarized in Fig. 4.

Figure 5 shows the classes of discrete convex functions we have introduced. L$^\natural$-convex functions contain L-convex functions as a special case. The same is true for M$^\natural$-convex and M-convex functions. By Theorems 3 and 6 both L$^\natural$-convex functions and M$^\natural$-convex functions are contained in the class of convex-extensible functions. It is known that the classes of L-convex functions and M-convex functions are disjoint, whereas the intersection of the classes of L$^\natural$-convex functions and M$^\natural$-convex functions is exactly the class of separable convex functions.

2.2.5 Discrete convex sets

In the continuous case the convexity of a set $S \subseteq \mathbf{R}^n$ can be characterized by that of its indicator function δ_S as

$$S \text{ is a convex set} \iff \delta_S \text{ is a convex function.}$$

We make use of this relation to define the concepts of discrete convex sets.

⟨Continuous Variables⟩ $f : \mathbf{R}^n \to \overline{\mathbf{R}}$		⟨Discrete Variables⟩ $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$
midpoint convex	$\longrightarrow$	discrete midpoint convex (L$^\natural$-convex)
$\Updownarrow$	[discretization]	
(ordinary) convex		
$\Updownarrow$	[discretization]	
equidistance convex	$\longrightarrow$	exchange property (M$^\natural$-convex)

discrete midpoint convex:	$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$
midpoint convex:	$f(x) + f(y) \geq 2 f\left(\frac{x+y}{2}\right)$
(ordinary) convex:	$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)$
equidistance convex:	$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y))$
exchange property:	$f(x) + f(y) \geq \min[f(x - \chi_i) + f(y + \chi_i), \min_{x_j < y_j}\{f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j)\}]$

Figure 4: Definitions of L$^\natural$-convexity and M$^\natural$-convexity by discretization

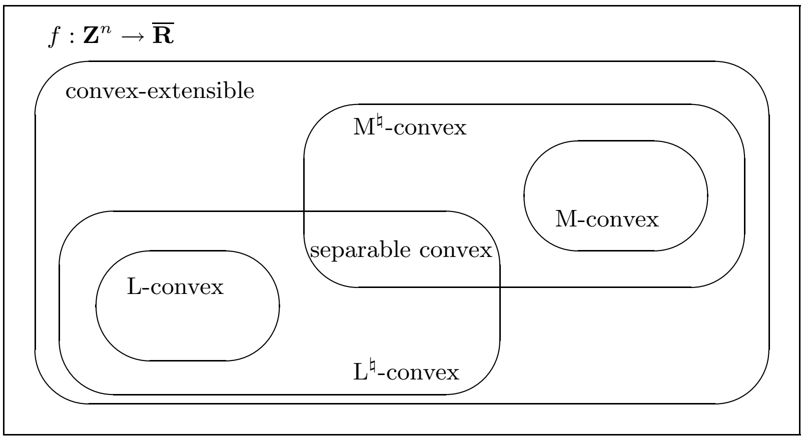

Figure 5: Classes of discrete convex functions
(L$^\natural$-convex $\cap$ M$^\natural$-convex = separable convex)

For a set $S \subseteq \mathbf{Z}^n$ the indicator function of S is a function $\delta_S : \mathbf{Z}^n \to \overline{\mathbf{R}}$ given by (2). *L$^\natural$-convex sets* and *M$^\natural$-convex sets* are defined as

$$
\begin{aligned}
S \text{ is an L}^\natural\text{-convex set} &\iff \delta_S \text{ is an L}^\natural\text{-convex function,} \\
S \text{ is an M}^\natural\text{-convex set} &\iff \delta_S \text{ is an M}^\natural\text{-convex function.}
\end{aligned}
$$

Similarly for the definitions of L-convex and M-convex sets. We have $S = \overline{S} \cap \mathbf{Z}^n$ for an L$^\natural$-convex (M$^\natural$-convex, L-convex or M-convex) set S, where $\overline{S}$ denotes the convex hull of S.

For an L$^\natural$-convex function f, both $\mathrm{dom}_{\mathbf{Z}} f$ and $\mathrm{argmin}_{\mathbf{Z}} f$ are L$^\natural$-convex sets. This statement remains true when L$^\natural$-convexity is replaced by M$^\natural$-convexity, L-convexity or M-convexity.

2.3 Discrete Convex Functions in Continuous Variables

So far we have been concerned with the translation from "continuous" to "discrete." We have defined L-convex and M-convex functions by discretization of midpoint convexity and equidistance convexity, respectively. Although these two properities are both equivalent to (ordinary) convexity for continuous functions in continuous variables, their discrete versions have given rise to different concepts (cf. Fig. 4).

We are now interested in the reverse direction, from "discrete" to "continuous," to define the concepts of L-convex and M-convex functions in continuous variables [50, 51, 52]. In so doing we intend to capture certain classes of convex functions with additional combinatorial structures. We refer to such functions as *discrete convex functions in continuous variables*. This may sound somewhat contradictory, but the adjective "discrete" indicates the discreteness in direction in the space $\mathbf{R}^n$ of continuous variables.

2.3.1 L-convex functions

L$^\natural$-convex functions in discrete variables have been introduced in terms of a discretization of midpoint convexity. By Theorem 5, however, we can alternatively say that L$^\natural$-convex functions are those functions which satisfy translation submodularity (9).

This alternative definition enables us to introduce the concept of L$^\natural$-convex functions in continuous variables. That is, a convex function $g : \mathbf{R}^n \to \overline{\mathbf{R}}$ is defined to be *L$^\natural$-convex* if

$$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \quad (\alpha \in \mathbf{R}_+,\ p, q \in \mathbf{R}^n), \tag{14}$$

where $\mathbf{R}_+$ denotes the set of nonnegative reals. Examples of L$^\natural$-convex functions are given in Section 4.1.

L$^\natural$-convex functions constitute a subclass of convex functions that are equipped with certain combinatorial properties in addition to convexity. It is known, for example, that a smooth function g is L$^\natural$-convex if and only if the Hessian matrix $H = (h_{ij} = \partial^2 g / \partial p_i \partial p_j)$ is a *diagonally dominant symmetric M-matrix*, i.e.,

$$h_{ij} \leq 0 \quad (i \neq j), \qquad \sum_{j=1}^{n} h_{ij} \geq 0 \quad (i = 1, \ldots, n) \tag{15}$$

at each point. This is a combinatorial property on top of positive semidefiniteness, which is familiar in operations research, mathematical economics, and numerical analysis. It may be said that L$^\natural$-convexity extends this well-known property to nonsmooth functions.

An *L-convex function* in continuous variables is defined as an L$^\natural$-convex function $g : \mathbf{R}^n \to \overline{\mathbf{R}}$ that satisfies

$$g(p + \alpha\mathbf{1}) = g(p) + \alpha r \qquad (\alpha \in \mathbf{R},\ p \in \mathbf{R}^n) \tag{16}$$

for some $r \in \mathbf{R}$ (which is independent of p and α). L-convex functions and L$^\natural$-convex functions are essentially the same, in that L$^\natural$-convex functions in n variables can be identified, up to the constant r in (16), with L-convex functions in $n + 1$ variables.

The inequality (14) is a continuous version of the translation submodularity (9), in which we had $\alpha \in \mathbf{Z}_+$ and $p, q \in \mathbf{Z}^n$ instead of $\alpha \in \mathbf{R}_+$ and $p, q \in \mathbf{R}^n$. It may be said that (14) is obtained from (9) by *prolongation*, by which we mean a process converse to discretization. Figure 6 summarizes how we have defined L$^\natural$-convex functions in discrete and continuous variables. Note that prolongation of discrete midpoint convexity renders no novel concept, but reduces to midpoint convexity, which is (almost) equivalent to convexity.

⟨Continuous Variables⟩		⟨Discrete Variables⟩
$g : \mathbf{R}^n \to \overline{\mathbf{R}}$		$g : \mathbf{Z}^n \to \overline{\mathbf{R}}$
(ordinary) convex		
$\Updownarrow$	[discretization]	
midpoint convex	$\longrightarrow$	discrete midpoint convex
		$\Updownarrow$
translation submodular	$\longleftarrow$	translation submodular
(L$^\natural$-convex)	[prolongation]	(L$^\natural$-convex)

(ordinary) convex:	$\lambda g(p) + (1-\lambda)g(q) \geq g(\lambda p + (1-\lambda)q)$
midpoint convex:	$g(p) + g(q) \geq 2\, g\left(\frac{p+q}{2}\right)$
discrete midpoint convex:	$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$
translation submodular:	$g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1}))$

Figure 6: Definitions of L$^\natural$-convexity by discretization and prolongation

2.3.2 M-convex functions

M$^\natural$-convex functions in continuous variables can be defined by prolongation (i.e., a continuous version) of the exchange property (13). We say that a convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is *M$^\natural$-convex* if, for any $x, y \in \mathrm{dom}_{\mathbf{R}} f$ and any $i \in \mathrm{supp}^+(x-y)$, there exist $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and a positive real number α_0 such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \tag{17}$$

for all $\alpha \in \mathbf{R}$ with $0 \leq \alpha \leq \alpha_0$.

M$^\natural$-convex functions in continuous variables constitute another subclass of convex functions, different from L$^\natural$-convex functions, that are equipped with another kind of combinatorial properties. See examples in Section 4.2.

An *M-convex function* in continuous variables is defined as an M$^\natural$-convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ that satisfies (17) with $j \in \mathrm{supp}^-(x-y)$. This is equivalent to saying that f is M-convex if and only if it is M$^\natural$-convex and $\mathrm{dom}_{\mathbf{R}} f \subseteq \{x \in \mathbf{R}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbf{R}$. M-convex functions and M$^\natural$-convex functions are essentially the same, in that M$^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n+1$ variables.

2.3.3 Classes of discrete convex functions in continuous variables

Figure 7 shows the classes of discrete convex functions in continuous variables. L$^\natural$-convex functions contain L-convex functions as a special case. The same is true for M$^\natural$-convex and M-convex functions. It is known that the classes of L-convex functions and M-convex functions are disjoint, whereas the intersection of the classes of L$^\natural$-convex functions and M$^\natural$-convex functions is exactly the class of separable convex functions.

Comparison of Fig. 7 with Fig. 5 shows the parallelism between the continuous and discrete cases.

3 Conjugacy

Conjugacy under the Legendre transformation is one of the most appealing facts in convex analysis. In discrete convex analysis, the discrete Legendre transformation gives a one-to-one correspondence between L-convex functions and M-convex functions.

3.1 Continuous Case

For a function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ (not necessarily convex) with $\mathrm{dom}_{\mathbf{R}} f \neq \emptyset$, the *convex conjugate* $f^\bullet : \mathbf{R}^n \to \overline{\mathbf{R}}$ is defined by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n), \tag{18}$$

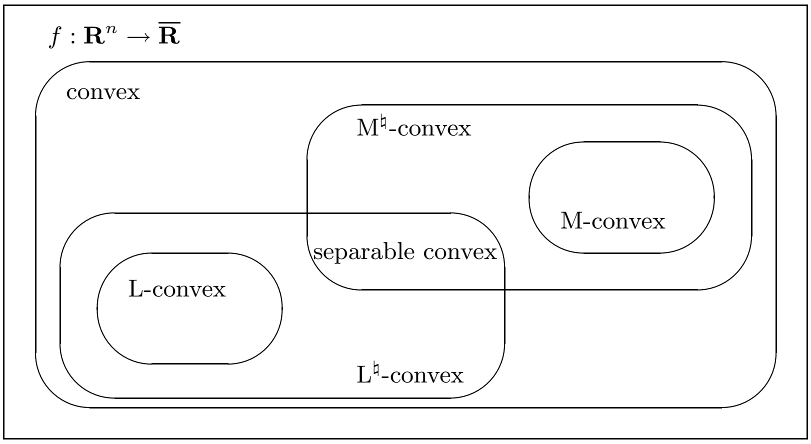

Figure 7: Classes of convex functions
(L$^\natural$-convex $\cap$ M$^\natural$-convex = separable convex)

where $\langle p, x \rangle = \sum_{i=1}^{n} p_i x_i$ is the inner product of $p = (p_i) \in \mathbf{R}^n$ and $x = (x_i) \in \mathbf{R}^n$. The function $f^\bullet$ is also referred to as the (convex) *Legendre(–Fenchel) transform* of f, and the mapping $f \mapsto f^\bullet$ as the (convex) *Legendre(–Fenchel) transformation*. Similarly to (18), the *concave conjugate* of $h : \mathbf{R}^n \to \underline{\mathbf{R}}$ is defined to be the function $h^\circ : \mathbf{R}^n \to \underline{\mathbf{R}}$ given by

$$h^\circ(p) = \inf\{\langle p, x \rangle - h(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n). \tag{19}$$

Note that $h^\circ(p) = -(-h)^\bullet(-p)$.

The conjugacy theorem in convex analysis states that the Legendre transformation gives a one-to-one correspondence in the class of closed proper convex functions, where a convex function f is said to be *proper* if $\mathrm{dom}_{\mathbf{R}} f$ is nonempty, and *closed* if the epigraph $\{(x, y) \in \mathbf{R}^{n+1} \mid y \geq f(x)\}$ is a closed subset of $\mathbf{R}^{n+1}$. Notation $f^{\bullet\bullet}$ means $(f^\bullet)^\bullet$.

Theorem 8. *The Legendre transformation* (18) *gives a symmetric one-to-one correspondence in the class of all closed proper convex functions. That is, for a closed proper convex function f, the conjugate function $f^\bullet$ is a closed proper convex function and $f^{\bullet\bullet} = f$.*

Addition of combinatorial ingredients to the above theorem yields the conjugacy between M-convex and L-convex functions.

Theorem 9 ([51])**.** *The Legendre transformation* (18) *gives a one-to-one correspondence between the classes of all closed proper M$^\natural$-convex functions and L$^\natural$-convex functions. Similarly for M-convex and L-convex functions.*

The first statement above means that, for a closed proper M$^\natural$-convex function f, $f^\bullet$ is a closed proper L$^\natural$-convex function and $f^{\bullet\bullet} = f$, and that, for a closed proper L$^\natural$-convex function g, $g^\bullet$ is a closed proper M$^\natural$-convex function and $g^{\bullet\bullet} = g$. To express this one-to-one correspondence we have indicated M$^\natural$-convex functions and L$^\natural$-convex functions by congruent regions in Fig. 7. The second statement means similarly that, for a closed proper M-convex function f, $f^\bullet$ is a closed proper L-convex function and $f^{\bullet\bullet} = f$, and that, for a closed proper L-convex function g, $g^\bullet$ is a closed proper M-convex function and $g^{\bullet\bullet} = g$. It is also noted that the conjugate of a separable convex function is another separable convex function.

The L/M-conjugacy is also valid for polyhedral convex functions.

Theorem 10 ([50])**.** *The Legendre transformation* (18) *gives a one-to-one correspondence between the classes of all polyhedral M$^\natural$-convex functions and L$^\natural$-convex functions. Similarly for M-convex and L-convex functions.*

3.2 Discrete Case

We turn to functions defined on integer points. For functions $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ and $h : \mathbf{Z}^n \to \underline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$ and $\mathrm{dom}_{\mathbf{Z}} h \neq \emptyset$, discrete versions of the Legendre transformations are defined by

$$
\begin{aligned}
f^{\bullet}(p) &= \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{R}^n), &(20)\\
h^{\circ}(p) &= \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{R}^n). &(21)
\end{aligned}
$$

We call (20) and (21), respectively, *convex* and *concave discrete Legendre(–Fenchel) transformations*. The functions $f^{\bullet} : \mathbf{R}^n \to \overline{\mathbf{R}}$ and $h^{\circ} : \mathbf{R}^n \to \underline{\mathbf{R}}$ are called the *convex conjugate* of f and the *concave conjugate* of h, respectively.

Theorem 11. *For an M$^{\natural}$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$, the conjugate function $f^{\bullet} : \mathbf{R}^n \to \overline{\mathbf{R}}$ is a (locally polyhedral) L$^{\natural}$-convex function. For an L$^{\natural}$-convex function $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$, the conjugate function $g^{\bullet} : \mathbf{R}^n \to \overline{\mathbf{R}}$ is a (locally polyhedral) M$^{\natural}$-convex function. Similarly for M-convex and L-convex functions.*

For an integer-valued function f, $f^{\bullet}(p)$ is integer for an integer vector p. Hence (20) with $p \in \mathbf{Z}^n$ defines a transformation of $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ to $f^{\bullet} : \mathbf{Z}^n \to \overline{\mathbf{Z}}$; we refer to (20) with $p \in \mathbf{Z}^n$ as $(20)_{\mathbf{Z}}$.

The conjugacy theorem for discrete M-convex and L-convex functions reads as follows.

Theorem 12 ([40])**.** *The discrete Legendre transformation* $(20)_{\mathbf{Z}}$ *gives a one-to-one correspondence between the classes of all integer-valued M$^{\natural}$-convex functions and L$^{\natural}$-convex functions in discrete variables. Similarly for M-convex and L-convex functions.*

It should be clear that the first statement above means that, for an integer-valued M$^{\natural}$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$, the function $f^{\bullet}$ in $(20)_{\mathbf{Z}}$ is an integer-valued L$^{\natural}$-convex function and $f^{\bullet\bullet} = f$, where $f^{\bullet\bullet}$ is a short-hand notation for $(f^{\bullet})^{\bullet}$ using the discrete Legendre transformation $(20)_{\mathbf{Z}}$, and similarly when f is L$^{\natural}$-convex.

4 Examples

4.1 L-convex Functions

Some examples of L$^{\natural}$- and L-convex functions are given in this section. The following basic facts are noted.

1. The effective domain of an L$^{\natural}$-convex function is an L$^{\natural}$-convex set.
2. An L$^{\natural}$-convex function remains to be L$^{\natural}$-convex when its effective domain is restricted to any L$^{\natural}$-convex set.
3. A sum of L$^{\natural}$-convex functions is L$^{\natural}$-convex.

Similar statements are true when "L$^{\natural}$-convex" is replaced by "L-convex" in the above.

We first consider functions in discrete variable $p = (p_1, \ldots, p_n) \in \mathbf{Z}^n$.

Linear function: A linear (or affine) function

$$g(p) = \alpha + \langle p, x\rangle \tag{22}$$

with $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ is L-convex (and hence L$^{\natural}$-convex).

Quadratic function: A quadratic function

$$g(p) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_i p_j \tag{23}$$

with $a_{ij} = a_{ji} \in \mathbf{R}$ $(i, j = 1, \ldots, n)$ is L$^{\natural}$-convex if and only if

$$a_{ij} \le 0 \quad (i \neq j), \qquad \sum_{j=1}^{n} a_{ij} \ge 0 \quad (i = 1, \ldots, n). \tag{24}$$

It is L-convex if and only if

$$a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_{j=1}^{n} a_{ij} = 0 \quad (i = 1, \ldots, n). \tag{25}$$

Separable convex function: For univariate convex functions ψ_i $(i = 1, \ldots, n)$ and ψ_{ij} $(i, j = 1, \ldots, n; i \neq j)$,

$$g(p) = \sum_{i=1}^{n} \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j) \tag{26}$$

is an L$^\natural$-convex function. This is L-convex if $\psi_i = 0$ for $i = 1, \ldots, n$.

Maximum-component function: For any $\tau_0, \tau_1, \ldots, \tau_n \in \underline{\mathbf{R}}$,

$$g(p) = \max\{\tau_0, p_1 + \tau_1,\ p_2 + \tau_2, \ldots,\ p_n + \tau_n\} \tag{27}$$

is an L$^\natural$-convex function. This is L-convex if τ_0 does not exist (i.e., $\tau_0 = -\infty$). Hence

$$g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\} \tag{28}$$

is an L-convex function. Furthermore, if ψ is a nondecreasing univariate convex function,

$$g(p) = \psi(\max_{1 \leq i \leq n}\{p_i + \tau_i\}) \tag{29}$$

is an L$^\natural$-convex function. It is also mentioned that, if $g_0(p, t)$ is L$^\natural$-convex in $(p, t) \in \mathbf{Z}^n \times \mathbf{Z}$ and nondecreasing in t, then the *max-aggregation* $g : \mathbf{Z}^n \times \mathbf{Z}^m \to \overline{\mathbf{R}}$ defined by

$$g(p, q) = g_0(p, \max(q_1, \ldots, q_m)) \qquad (p \in \mathbf{Z}^n, q \in \mathbf{Z}^m) \tag{30}$$

is L$^\natural$-convex in (p, q), whereas g is L-convex if g_0 is L-convex.

Submodular set function: A submodular set function $\rho : 2^V \to \overline{\mathbf{R}}$ can be identified with an L$^\natural$-convex function g under the correspondence $g(\chi_X) = \rho(X)$ for $X \subseteq V$, where $\mathrm{dom}_{\mathbf{Z}} g \subseteq \{0, 1\}^n$.

Multimodular function: A function $h : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is *multimodular* if and only if it can be represented as

$$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$$

for some L$^\natural$-convex function g; see [1, 2, 22, 45].

The constructions above work for functions in continuous variable $p \in \mathbf{R}^n$. That is, the functions $g : \mathbf{R}^n \to \overline{\mathbf{R}}$ defined by the expressions (22) to (30) are L$^\natural$- or L-convex functions, if all the variables are understood as real numbers or vectors. It is noteworthy that quadratic L$^\natural$-convex functions are exactly the same as the (finite dimensional case of) *Dirichlet forms* used in probability theory [21]. The energy consumed in a nonlinear electrical network, when expressed as a function in terminal voltages, is an L$^\natural$-convex function [43, Section 2.2].

4.2 M-convex Functions

Some examples of M$^\natural$- and M-convex functions are given in this section. The following basic facts are noted.

1. The effective domain of an M$^\natural$-convex function is an M$^\natural$-convex set.

2. An M$^\natural$-convex function does not necessarily remain M$^\natural$-convex when its effective domain is restricted to an M$^\natural$-convex set.

3. A sum of M$^\natural$-convex functions is not necessarily M$^\natural$-convex.

4. The *infimal convolution* of M$^\natural$-convex functions f_1 and f_2, defined as

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x = x_1 + x_2\}, \tag{31}$$

   is M$^\natural$-convex if $f_1 \Box f_2$ does not take $-\infty$, where $x_1, x_2 \in \mathbf{Z}^n$ in the discrete case and $x_1, x_2 \in \mathbf{R}^n$ in the continuous case.

Similar statements are true when "M$^\natural$-convex" is replaced by "M-convex" in the above.

We first consider functions in discrete variable $x = (x_1, \ldots, x_n) \in \mathbf{Z}^n$.

Linear function: A linear (or affine) function

$$f(x) = \alpha + \langle p, x \rangle \tag{32}$$

with $p \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ is M$^\natural$-convex. It is M-convex if $\mathrm{dom}_{\mathbf{Z}} f$ is an M-convex set.

Quadratic function: A quadratic function

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{33}$$

with $a_{ij} = a_{ji} \in \mathbf{R}$ $(i, j = 1, \ldots, n)$ is M$^\natural$-convex if and only if $a_{ij} \geq 0$ for all (i, j) and

$$a_{ij} \geq \min(a_{ik}, a_{jk}) \ \text{ if } \ \{i, j\} \cap \{k\} = \emptyset, \tag{34}$$

where $\mathrm{dom}_{\mathbf{Z}} f = \mathbf{Z}^n$. A function f of (33), with $\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^n \mid \sum_{i=1}^{n} x_i = r\}$ for some $r \in \mathbf{Z}$, is M-convex if and only if

$$a_{ij} + a_{kl} \geq \min(a_{ik} + a_{jl}, a_{il} + a_{jk}) \ \text{ if } \ \{i, j\} \cap \{k, l\} = \emptyset. \tag{35}$$

Laminar convex function: By a *laminar family* we mean a nonempty family $\mathcal{T}$ of subsets of V such that $X \cap Y = \emptyset$ or $X \subseteq Y$ or $X \supseteq Y$ for any $X, Y \in \mathcal{T}$. A function f is called *laminar convex* if it can be represented as

$$f(x) = \sum_{X \in \mathcal{T}} f_X(x(X)) \tag{36}$$

for a laminar family $\mathcal{T}$ and a family of univariate convex functions f_X indexed by $X \in \mathcal{T}$, where $x(X) = \sum_{i \in X} x_i$. A laminar convex function is M$^\natural$-convex. A separable convex function (3) is laminar convex and hence M$^\natural$-convex. It is known [23] that every quadratic M$^\natural$-convex function (in discrete variables) is laminar convex.

Minimum-value function: Given a_i for $i \in V$ we define a set function $\mu : 2^V \to \overline{\mathbf{R}}$ as $\mu(X) = \min\{a_i \mid i \in X\}$ for nonempty $X \subseteq V$. By convention we put $\mu(\emptyset) = a_*$ by choosing $a_* \in \overline{\mathbf{R}}$ such that $a_* \geq \max\{a_i \mid i \in V\}$. Then μ is M$^\natural$-convex when identified with a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{Z}} f \subseteq \{0, 1\}^n$ by $f(\chi_X) = \mu(X)$ for $X \subseteq V$.

Bipartite matching: Let $G = (V, W; E)$ be a bipartite graph with vertex set $V \cup W$ and edge set E, and suppose that each edge $e \in E$ is associated with weight $\gamma(e) \in \mathbf{R}$. For $X \subseteq V$ denote by $\Gamma(X)$ the minimum weight of a matching that matches with X, i.e.,

$$\Gamma(X) = \min\{\sum_{e \in M} \gamma(e) \mid M \text{ is a matching, } V \cap \partial M = X\},$$

where $\Gamma(X) = +\infty$ if such M does not exist. Then Γ is M$^\natural$-convex when identified with a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{Z}} f \subseteq \{0, 1\}^n$ by $f(\chi_X) = \Gamma(X)$ for $X \subseteq V$. This construction can be extended to the minimum convex-cost flow problem.

Stable marriage problem: The payoff function of the stable marriage problem is M$^\natural$-concave [20, 68].

Matroid: Let $(V, \mathcal{B}, \mathcal{I}, \rho)$ be a matroid on V with base family $\mathcal{B}$, independent-set family $\mathcal{I}$ and rank function ρ. The characteristic vectors of bases $\{\chi_B \mid B \in \mathcal{B}\}$ form an M-convex set and those of independent sets $\{\chi_I \mid I \in \mathcal{I}\}$ form an M$^\natural$-convex set. The rank function $\rho : 2^V \to \mathbf{Z}$ is M$^\natural$-concave when identified with a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}_{\mathbf{Z}} f = \{0, 1\}^n$ by $f(\chi_X) = \rho(X)$ for $X \subseteq V$; see Section 6.1. More generally, the *vector rank function* of an integral submodular system is M$^\natural$-concave [18, p. 51].

Valuated matroid: A *valuated matroid* $\omega : 2^V \to \underline{\mathbf{R}}$ of [9, 10] (see also [42, Chapter 5]) can be identified with an M$^\natural$-concave function f under the correspondence $f(\chi_X) = \omega(X)$ for $X \subseteq V$, where $\mathrm{dom}_{\mathbf{Z}} f \subseteq \{0, 1\}^n$. The tropical geometry [67] is closely related with valuated matroids. For example, the tropical linear space [66] is essentially the same as the circuit valuation of matroids [54].

Next we turn to functions $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ in continuous variable $x \in \mathbf{R}^n$. The infimal convolution (31) preserves M$^\natural$-convexity when the infimum is taken over $x_1, x_2 \in \mathbf{R}^n$. Laminar convex functions (36) as well as linear functions (32) remain to be M$^\natural$-convex when x is understood as a real vector. The energy consumed in a nonlinear electrical network, when expressed as a function in terminal currents, is an M$^\natural$-convex function [43, Section 2.2].

A subtlety arises for quadratic functions. Condition (34), together with $a_{ij} \geq 0$ for all (i, j), is sufficient but not necessary for $f : \mathbf{R}^n \to \mathbf{R}$ of the form of (33) to be M$^\natural$-convex. A necessary and sufficient condition in terms of the matrix $A = (a_{ij})$ is that, for any $\beta > 0$, $A + \beta I$ is nonsingular and $(A + \beta I)^{-1}$ satisfies (24). It is also mentioned that not every quadratic M$^\natural$-convex function in real variables is laminar convex. As for M-convexity, condition (35) is sufficient but not necessary for f to be M-convex.

Thus the relation between discrete and continuous cases are not so simple in M-convexity as in L-convexity.

5 Separation and Fenchel Duality

5.1 Separation Theorem

The *duality* principle in convex analysis can be expressed in a number of different forms. One of the most appealing statements is in the form of the separation theorem, which asserts the existence of a separating affine function $y = \alpha^* + \langle p^*, x\rangle$ for a pair of convex and concave functions.

In the continuous case we have the following.

Theorem 13. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ and $h : \mathbf{R}^n \to \underline{\mathbf{R}}$ be convex and concave functions, respectively (satisfying certain regularity conditions). If*

$$f(x) \geq h(x) \qquad (\forall x \in \mathbf{R}^n),$$

there exist $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^n$ such that

$$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x) \qquad (\forall x \in \mathbf{R}^n).$$

A *discrete separation theorem* means a statement like:

> For any $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ and $h : \mathbf{Z}^n \to \underline{\mathbf{R}}$ belonging to certain classes of functions, if $f(x) \geq h(x)$ for all $x \in \mathbf{Z}^n$, then there exist $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^n$ such that
>
> $$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x) \qquad (\forall x \in \mathbf{Z}^n).$$
>
> Moreover, if f and h are integer-valued, there exist integer-valued $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^n$.

Discrete separation theorems often capture deep combinatorial properties in spite of the apparent similarity to the separation theorem in convex analysis. In this connection we note the following facts (see [43, Examples 1.5 and 1.6] for concrete examples), where $\overline{f}$ denotes the convex closure of f, $\overline{h}$ the concave closure of h, and $\not\Longrightarrow$ stands for "does not imply."

1. $f(x) \geq h(x)$ $(\forall x \in \mathbf{Z}^n)$ $\not\Longrightarrow$ $\overline{f}(x) \geq \overline{h}(x)$ $(\forall x \in \mathbf{R}^n)$.
2. $f(x) \geq h(x)$ $(\forall x \in \mathbf{Z}^n)$ $\not\Longrightarrow$ existence of $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^n$.
3. existence of $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^n$ $\not\Longrightarrow$ existence of $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^n$.

The separation theorems for M-convex/M-concave functions and for L-convex/L-concave functions read as follows. It should be clear that $f^\bullet$ and h° are the convex and concave conjugate functions of f and h defined by (20) and (21), respectively.

Theorem 14 (M-separation theorem)**.** *Let $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an M$^\natural$-convex function and $h : \mathbf{Z}^n \to \underline{\mathbf{R}}$ be an M$^\natural$-concave function such that* $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} h \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{R}} f^\bullet \cap \mathrm{dom}_{\mathbf{R}} h^\circ \neq \emptyset$*. If $f(x) \geq h(x)$ $(\forall x \in \mathbf{Z}^n)$, there exist $\alpha^* \in \mathbf{R}$ and $p^* \in \mathbf{R}^n$ such that*

$$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x) \qquad (\forall x \in \mathbf{Z}^n).$$

Moreover, if f and h are integer-valued, there exist integer-valued $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^n$.

Theorem 15 (L-separation theorem)**.** *Let $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an $L^\natural$-convex function and $k : \mathbf{Z}^n \to \underline{\mathbf{R}}$ be an $L^\natural$-concave function such that $\mathrm{dom}_{\mathbf{Z}} g \cap \mathrm{dom}_{\mathbf{Z}} k \neq \emptyset$ or $\mathrm{dom}_{\mathbf{R}} g^\bullet \cap \mathrm{dom}_{\mathbf{R}} k^\circ \neq \emptyset$. If $g(p) \geq k(p)$ $(\forall p \in \mathbf{Z}^n)$, there exist $\beta^* \in \mathbf{R}$ and $x^* \in \mathbf{R}^n$ such that*

$$g(p) \geq \beta^* + \langle p, x^* \rangle \geq k(p) \qquad (\forall p \in \mathbf{Z}^n).$$

Moreover, if g and k are integer-valued, there exist integer-valued $\beta^* \in \mathbf{Z}$ and $x^* \in \mathbf{Z}^n$.

As an immediate corollary of the M-separation theorem we can obtain an optimality criterion for the problem of minimizing the sum of two M-convex functions, which we call the *M-convex intersection problem*. Note that the sum of M-convex functions is no longer M-convex and Theorem 7 does not apply.

Theorem 16 (M-convex intersection theorem)**.** *For $M^\natural$-convex functions $f_1, f_2 : \mathbf{Z}^n \to \overline{\mathbf{R}}$ and a point $x^* \in \mathrm{dom}_{\mathbf{Z}} f_1 \cap \mathrm{dom}_{\mathbf{Z}} f_2$ we have*

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \qquad (\forall x \in \mathbf{Z}^n)$$

if and only if there exists $p^* \in \mathbf{R}^n$ such that

$$\begin{aligned} (f_1 - p^*)(x^*) &\leq (f_1 - p^*)(x) && (\forall x \in \mathbf{Z}^n), \\ (f_2 + p^*)(x^*) &\leq (f_2 + p^*)(x) && (\forall x \in \mathbf{Z}^n). \end{aligned}$$

These conditions are equivalent, respectively, to

$$\begin{aligned} (f_1 - p^*)(x^*) &\leq (f_1 - p^*)(x^* + \chi_i - \chi_j) && (\forall\, i, j \in \{0, 1, \ldots, n\}), \\ (f_2 + p^*)(x^*) &\leq (f_2 + p^*)(x^* + \chi_i - \chi_j) && (\forall\, i, j \in \{0, 1, \ldots, n\}), \end{aligned}$$

and for such p^* we have

$$\mathrm{argmin}_{\mathbf{Z}}(f_1 + f_2) = \mathrm{argmin}_{\mathbf{Z}}(f_1 - p^*) \cap \mathrm{argmin}_{\mathbf{Z}}(f_2 + p^*).$$

Moreover, if f_1 and f_2 are integer-valued, we can choose integer-valued $p^* \in \mathbf{Z}^n$.

Frank's discrete separation theorem [16] for submodular/supermodular set functions is a special case of the L-separation theorem. Frank's weight splitting theorem [15] for the weighted matroid intersection problem is a special case of the M-convex intersection problem. The submodular flow problem can be generalized to the M-convex submodular flow problem [41]; see also [25, 26].

5.2 Fenchel Duality

Another expression of the duality principle is in the form of the Fenchel duality. This is a min-max relation between a pair of convex and concave functions and their conjugate functions. Such a min-max theorem is computationally useful in that it affords a certificate of optimality.

The Fenchel duality theorem in the continuous case reads as follows. Recall the notations $f^\bullet$ and h° in (18) and (19).

Theorem 17. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ and $h : \mathbf{R}^n \to \underline{\mathbf{R}}$ be convex and concave functions, respectively (satisfying certain regularity conditions). Then*

$$\inf\{f(x) - h(x) \mid x \in \mathbf{R}^n\} = \sup\{h^\circ(p) - f^\bullet(p) \mid p \in \mathbf{R}^n\}.$$

We now turn to the discrete case. For any functions $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ and $h : \mathbf{Z}^n \to \underline{\mathbf{Z}}$ we have a chain of inequalities:

$$\begin{array}{ccc} \inf\{f(x) - h(x) \mid x \in \mathbf{Z}^n\} & & \sup\{h^\circ(p) - f^\bullet(p) \mid p \in \mathbf{Z}^n\} \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\geq$}} \\ \inf\{\overline{f}(x) - \overline{h}(x) \mid x \in \mathbf{R}^n\} & \geq & \sup\{\overline{h}^\circ(p) - \overline{f}^\bullet(p) \mid p \in \mathbf{R}^n\} \end{array} \tag{37}$$

from the definitions (20) and (21) of conjugate functions $f^\bullet$ and h°, where $\overline{f}$ and $\overline{h}$ are convex and concave closures of f and h, respectively. It should be observed that

1. The second inequality in the middle of (37) is in fact an equality by the Fenchel duality theorem (Theorem 17) in convex analysis;

2. The first (left) inequality in (37) can be strict even when f is convex-extensible and h is concave-extensible, and similarly for the third (right) inequality. See Examples 1 and 2 below.

Example 1. For $f, h : \mathbf{Z}^2 \to \mathbf{Z}$ defined as

$$f(x_1, x_2) = |x_1 + x_2 - 1|, \qquad h(x_1, x_2) = 1 - |x_1 - x_2|$$

we have $\inf\{f - h\} = 0$, $\inf\{\overline{f} - \overline{h}\} = -1$. The discrete Legendre transforms are given by

$$f^{\bullet}(p_1, p_2) = \begin{cases} p_1 & ((p_1, p_2) \in S) \\ +\infty & \text{(otherwise)}, \end{cases} \qquad h^{\circ}(p_1, p_2) = \begin{cases} -1 & ((p_1, p_2) \in T) \\ -\infty & \text{(otherwise)} \end{cases}$$

with $S = \{(-1, -1), (0, 0), (1, 1)\}$ and $T = \{(-1, 1), (0, 0), (1, -1)\}$. Hence $\sup\{h^{\circ} - f^{\bullet}\} = h^{\circ}(0, 0) - f^{\bullet}(0, 0) = -1 - 0 = -1$. Then (37) reads as

$$\begin{array}{ccccccc} \inf\{f - h\} & > & \inf\{\overline{f} - \overline{h}\} & = & \sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\} & = & \sup\{h^{\circ} - f^{\bullet}\}. \\ (0) & & (-1) & & (-1) & & (-1) \end{array}$$

■

Example 2. For $f, h : \mathbf{Z}^2 \to \mathbf{Z}$ defined as

$$f(x_1, x_2) = \max(0, x_1 + x_2), \qquad h(x_1, x_2) = \min(x_1, x_2)$$

we have $\inf\{f - h\} = \inf\{\overline{f} - \overline{h}\} = 0$. The discrete Legendre transforms are given as $f^{\bullet} = \delta_S$ and $h^{\circ} = -\delta_T$ in terms of the indicator functions of $S = \{(0, 0), (1, 1)\}$ and $T = \{(1, 0), (0, 1)\}$. Since $S \cap T = \emptyset$, $h^{\circ} - f^{\bullet}$ is identically equal to $-\infty$, whereas $\sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\} = 0$ since $\overline{f}^{\bullet} = \delta_{\overline{S}}$, $\overline{h}^{\circ} = -\delta_{\overline{T}}$ and $\overline{S} \cap \overline{T} = \{(1/2, 1/2)\}$. Then (37) reads as

$$\begin{array}{ccccccc} \inf\{f - h\} & = & \inf\{\overline{f} - \overline{h}\} & = & \sup\{\overline{h}^{\circ} - \overline{f}^{\bullet}\} & > & \sup\{h^{\circ} - f^{\bullet}\}. \\ (0) & & (0) & & (0) & & (-\infty) \end{array}$$

■

From the observations above, we see that the essence of the following theorem is the assertion that the first and third inequalities in (37) are in fact equalities for M$^{\natural}$-convex/M$^{\natural}$-concave functions and L$^{\natural}$-convex/L$^{\natural}$-concave functions.

Theorem 18 (Fenchel-type duality theorem)**.**
(1) *Let $f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ be an integer-valued M$^{\natural}$-convex function and $h : \mathbf{Z}^n \to \underline{\mathbf{Z}}$ be an integer-valued M$^{\natural}$-concave function such that* $\mathrm{dom}_{\mathbf{Z}} f \cap \mathrm{dom}_{\mathbf{Z}} h \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} h^{\circ} \neq \emptyset$*. Then we have*

$$\inf\{f(x) - h(x) \mid x \in \mathbf{Z}^n\} = \sup\{h^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbf{Z}^n\}. \tag{38}$$

If this common value is finite, the infimum and the supremum are attained.
(2) *Let $g : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ be an integer-valued L$^{\natural}$-convex function and $k : \mathbf{Z}^n \to \underline{\mathbf{Z}}$ be an integer-valued L$^{\natural}$-concave function such that* $\mathrm{dom}_{\mathbf{Z}} g \cap \mathrm{dom}_{\mathbf{Z}} k \neq \emptyset$ *or* $\mathrm{dom}_{\mathbf{Z}} g^{\bullet} \cap \mathrm{dom}_{\mathbf{Z}} k^{\circ} \neq \emptyset$*. Then we have*

$$\inf\{g(p) - k(p) \mid p \in \mathbf{Z}^n\} = \sup\{k^{\circ}(x) - g^{\bullet}(x) \mid x \in \mathbf{Z}^n\}. \tag{39}$$

If this common value is finite, the infimum and the supremum are attained.

Edmonds' intersection theorem [11] in the integral case is a special case of Theorem 18 (1) above, and Fujishige's Fenchel-type duality theorem [17] (see also [18, Section 6.1]) for submodular set functions is a special case of Theorem 18 (2) above.

Whereas L-separation and M-separation theorems are parallel or conjugate in their statements, the Fenchel-type duality theorem is self-conjugate, in that the substitution of $f = g^{\bullet}$ and $h = k^{\circ}$ into (38) results in (39) by virtue of $g = g^{\bullet\bullet}$ and $k = k^{\circ\circ}$. With the knowledge of M-/L-conjugacy, these three duality theorems are almost equivalent to one another; once one of them is established, the other two theorems can be derived by relatively easy formal calculations.

6 Submodular Function Maximization

Maximization of a submodular set function is a difficult task in general. Many NP-hard problems can be reduced to this problem. Also known is that no polynomial algorithm exists in the ordinary oracle model (and this statement is independent of the P≠NP conjecture) [27, 33, 34]. For approximate maximization under matroid constraints the performance bounds of greedy or ascent type algorithms were analyzed in [7, 14, 56]. See, e.g., [4, 5, 6, 13] for recent development.

M$^\natural$-concave functions on $\{0, 1\}$-vectors form a subclass of submodular set functions that are algorithmically tractable for maximization [48]. This is compatible with our general understanding that concave functions are easy to maximize, and explains why certain submodular functions treated in the literature are easier to maximize. To be specific, we have the following.

1. The greedy algorithm can be generalized for maximization of a single M$^\natural$-concave function.
2. The matroid intersection algorithm can be generalized for maximization of a sum of two M$^\natural$-concave functions.

Note that a sum of M$^\natural$-concave functions is not necessarily M$^\natural$-concave, though it is submodular. It is also mentioned that maximization of a sum of three M$^\natural$-concave functions is NP-hard, since it includes the three-matroid intersection problem as a special case.

6.1 M$^\natural$-concave set functions

Let us say that a set function $\rho : 2^V \to \mathbf{R}$ is *M$^\natural$-concave* if the function $h : \mathbf{Z}^n \to \underline{\mathbf{R}}$ defined as $h(\chi_X) = \rho(X)$ for $X \subseteq V$ and $h(x) = -\infty$ for $x \notin \{0, 1\}^n$ is M$^\natural$-concave. In other words, ρ is M$^\natural$-concave if and only if, for any $X, Y \subseteq V$ and $i \in X \setminus Y$, we have $\rho(X) + \rho(Y) \leq \rho(X \setminus \{i\}) + \rho(Y \cup \{i\})$ or $\rho(X) + \rho(Y) \leq \rho((X \setminus \{i\}) \cup \{j\}) + \rho((Y \cup \{i\}) \setminus \{j\})$ for some $j \in Y \setminus X$. An M$^\natural$-concave set function is submodular [43, Theorem 6.19].

Not every submodular set function is M$^\natural$-concave. An example of a submodular function that is not M$^\natural$-concave is given by ρ on $V = \{1, 2, 3\}$ defined as $\rho(\emptyset) = 0$, $\rho(\{2, 3\}) = 2$, $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = \rho(\{1, 2\}) = \rho(\{1, 3\}) = \rho(\{1, 2, 3\}) = 1$. The condition above fails for $X = \{2, 3\}$, $Y = \{1\}$ and $i = 2$.

A simple example of an M$^\natural$-concave set function is given by $\rho(X) = \varphi(|X|)$, where φ is a univariate concave function. This is a classical example [34] of a submodular function that connects submodularity and concavity.

For a family of univariate concave functions $\{\varphi_A \mid A \in \mathcal{T}\}$ indexed by a family $\mathcal{T}$ of subsets of V, the function

$$\rho(X) = \sum_{A \in \mathcal{T}} \varphi_A(|A \cap X|) \qquad (X \subseteq V)$$

is submodular. This function is M$^\natural$-concave if, in addition, $\mathcal{T}$ is a laminar family (i.e., $A, B \in \mathcal{T} \Rightarrow A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$).

Given a set of real numbers a_i indexed by $i \in V$, the *maximum-value function*

$$\rho(X) = \max_{i \in X} a_i \qquad (X \subseteq V)$$

is an M$^\natural$-concave function, where $\rho(\emptyset)$ is defined to be sufficiently small.

A *matroid rank function* is M$^\natural$-concave. Given a matroid on V in terms of the family $\mathcal{I}$ of independent sets, the rank function ρ is defined by

$$\rho(X) = \max\{|I| \mid I \in \mathcal{I},\ I \subseteq X\} \qquad (X \subseteq V),$$

which denotes the maximum size of an independent set contained in X. An interesting identity exists that indicates a kind of self-conjugacy of a matroid rank function. Let $g : \mathbf{Z}^n \to \overline{\mathbf{Z}}$ be such that $g(\chi_X) = \rho(X)$ for $X \subseteq V$ and $\mathrm{dom}_{\mathbf{Z}} g = \{0, 1\}^n$, and denote by $\rho^\bullet$ the discrete Legendre transform $g^\bullet$ of g defined by $(20)_{\mathbf{Z}}$ (i.e., (20) with $p \in \mathbf{Z}^n$). Then we have

$$\rho(X) = |X| - \rho^\bullet(\chi_X) \qquad (X \subseteq V). \tag{40}$$

This can be shown as follows: $\rho^\bullet(\chi_X) = \max_Y\{|X \cap Y| - \rho(Y) \mid Y \subseteq V\} = \max_Y\{|X \cap Y| - \rho(Y) \mid X \subseteq Y \subseteq V\} = \max_Y\{|X| - \rho(Y) \mid X \subseteq Y \subseteq V\} = |X| - \rho(X)$; see also [18, Lemma 6.2]. Since ρ is submodular, g is L$^\natural$-convex, and hence $g^\bullet$ $(= \rho^\bullet)$ is M$^\natural$-convex by conjugacy (Theorem 12). Then the expression (40) shows that ρ is M$^\natural$-concave.

A *weighted matroid rank function*, represented as

$$\rho(X) = \max\{\sum_{i \in I} w_i \mid I \in \mathcal{I},\ I \subseteq X\} \qquad (X \subseteq V) \tag{41}$$

with a nonnegative vector $w \in \mathbf{R}^n$, is also M$^\natural$-concave [62, 63]. It is noted that a polymatroid rank function is not necessarily M$^\natural$-concave.

6.2 Greedy algorithm

M$^\natural$-concave set functions admit the following local characterization of global maximum, an immediate corollary of Theorem 7.

Theorem 19. *For an M$^\natural$-concave set function $\rho : 2^V \to \mathbf{R}$ and a subset $X \subseteq V$, we have $\rho(X) \geq \rho(Y)$ $(\forall\, Y \subseteq V)$ if and only if*

$$\rho(X) \geq \max_{i \in X, j \in V \setminus X} \{\rho((X \setminus \{i\}) \cup \{j\}),\ \rho(X \setminus \{i\}),\ \rho(X \cup \{j\})\}.$$

A natural greedy algorithm works for maximization of an M$^\natural$-concave set function ρ:

S0: Put $X := \emptyset$.
S1: Find $j \in V \setminus X$ that maximizes $\rho(X \cup \{j\})$.
S2: If $\rho(X) \geq \rho(X \cup \{j\})$, then stop ($X$ is a maximizer of ρ).
S3: Set $X := X \cup \{j\}$ and go to S1.

This algorithm may be regarded as a variant of the algorithm of Dress–Wenzel [9] for valuated matroids, and the validity can be shown similarly.

6.3 Intersection algorithm

Edmonds's matroid intersection/union algorithms show that we can efficiently maximize $\rho_1(X) + \rho_2(V \setminus X)$ and $\rho_1(X) + \rho_2(X) - |X|$ for two matroid rank functions ρ_1 and ρ_2. It should be clear that $\max_X\{\rho_1(X) + \rho_2(V \setminus X)\}$ is equal to the rank of the union of two matroids (V, ρ_1) and (V, ρ_2), and that $\max_X\{\rho_1(X) + \rho_2(X) - |X|\}$ is equal to the maximum size of a common independent set for matroid (V, ρ_1) and the dual of matroid (V, ρ_2). We note here that both $\rho_1(X) + \rho_2(V \setminus X)$ and $\rho_1(X) + (\rho_2(X) - |X|)$ are submodular functions that are represented as a sum of two M$^\natural$-concave functions.

Edmonds's intersection algorithm can be generalized for M$^\natural$-concave functions. A sum of two M$^\natural$-concave set functions can be maximized in polynomial time by means of a variant of the valuated matroid intersection algorithm [37, 38]; see also [41, 42, 43]. It follows from the M-convex intersection theorem (Theorem 16) that, for two M$^\natural$-concave set functions ρ_1 and ρ_2, X maximizes $\rho_1(X) + \rho_2(X)$ if and only if there exists $p^* \in \mathbf{R}^n$ such that X maximizes both $\rho_1(X) + p^*(X)$ and $\rho_1(X) - p^*(X)$ at the same time, where $p^*(X) = \sum_{i \in X} p_i^*$.

Conclusion

Efficient algorithms are available for minimization of L-convex and M-convex functions [43, Chapter 10]. The complexity analysis for the L-convex function minimization algorithm of [44] is improved in [31, 53]. We also refer to [61, 69] for M-convex function minimization, and [25] for the submodular flow problem, or equivalently for the Fenchel duality. Most of the efficient algorithms employ scaling techniques based on proximity theorems; see [26, 36, 55] for proximity theorems.

Discrete convex functions appear naturally in operations research. Multimodular functions, which are L$^\natural$-convex functions in disguise, are used in queueing theory [1, 2, 22, 45] and inventory theory [65, 72].

A jump system [3] is a generalization of a matroid, a delta-matroid and a base polyhedron of an integral polymatroid (or a submodular system). The concept of M-convex functions can be extended to functions on constant-parity jump systems [46]. For $x, y \in \mathbf{Z}^n$ we call $s \in \mathbf{Z}^n$ an (x, y)-increment if $s = \chi_i$ for some $i \in \mathrm{supp}^+(y - x)$ or $s = -\chi_i$ for some $i \in \mathrm{supp}^-(y - x)$. We call $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ an M-convex function (on a constant-parity jump system) if it satisfies the following exchange property: For any $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ and any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

It then follows that $\mathrm{dom}_{\mathbf{Z}} f$ is a constant-parity jump system. Theorem 7 can be extended and operations such as infimal convolution can be generalized. See [28, 29, 30, 64].

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