研 究 速 報

English translation: 連続/離散ハイブリッド M$^\natural$ 凸関数に関する一考察*

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On Continuous/Discrete Hybrid M$^\natural$-Convex Functions*

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1. Introduction

The algorithm which solves a mixed integer quadratic problem quickly is required recently in the filed of model predictive control for hybrid systems [1]. In this context, Takamatsu, Hara, and Murota [8] proposed a concept of continuous/discrete hybrid convexity. That work proposed a class of continuous/discrete hybrid L$^\natural$-convex functions based on discrete convex analysis [4,5] and showed its minimality criterion. This paper, as additional considerations on hybrid convexity, proposes continuous/discrete hybrid M$^\natural$-convex functions and shows an optimality criterion for an integral polyhedral hybrid M$^\natural$-convex function minimization. It is also a purpose of this paper to point out that hybrid M$^\natural$-convexity, compared with hybrid L$^\natural$-convexity, needs more delicate theoretical treatment.

2. Preliminary

We define an M$^\natural$-convex function over integer lattice points, a polyhedral M$^\natural$-convex function, and a closed proper M$^\natural$-convex function as follows [4,5,7]. A function $f:\mathbf{Z}^n \to \mathbf{R}\cup\{+\infty\}$ is said to be M$^\natural$-convex if it satisfies the following (M$^\natural$-EXC[$\mathbf{Z}$]):

(M$^\natural$-EXC[Z]) For $x,y\in \mathrm{dom}_{\mathbf{Z}} f$ and $u\in \mathrm{supp}^+(x-y)$, there exist $v\in\mathrm{supp}^-(x-y)\cup\{0\}$ such that

$$f(x)+f(y)\geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v).$$

Here, we put $V=\{1,\ldots,n\}$, let $\chi_u\in\{0,1\}^n$ denote the characteristic vector of $u\in V$ and $\chi_0=(0,0,\ldots,0)$, and define $\mathrm{dom}_{\mathbf{Z}} f=\{x\in\mathbf{Z}^n \mid f(x)<+\infty\}$, $\mathrm{supp}^+(x-y)=\{u\in V\mid x(u)>y(u)\}$, $\mathrm{supp}^-(x-y)=\{u\in V\mid x(u)<y(u)\}$. A polyhedral function $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ with $\mathrm{dom}_{\mathbf{R}} f=\{x\in\mathbf{R}^n\mid f(x)<+\infty\}\neq\emptyset$ is said to be polyhedral M$^\natural$-convex if it satisfies (M$^\natural$-EXC[$\mathbf{R}$]):

(M$^\natural$-EXC[R]) For $x,y\in\mathrm{dom}_{\mathbf{R}} f$ and $u\in\mathrm{supp}^+(x-y)$, there exists $v\in\mathrm{supp}^-(x-y)\cup\{0\}$ and positive number $\alpha_0\in\mathbf{R}$ such that

$$f(x)+f(y)\geq f(x-\alpha(\chi_u-\chi_v))+f(y+\alpha(\chi_u-\chi_v))$$

for all $\alpha\in[0,\alpha_0]$.

Let $f:\mathbf{Z}^n\to\mathbf{R}\cup\{+\infty\}$ be a function defined on the integer lattice, where it is a tacit agreement that the effective domain $\mathrm{dom}_{\mathbf{Z}} f$ is nonempty. The convex closure of f is defined to be a function $\bar{f}:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty,-\infty\}$ given by

$$\bar{f}(x)=\sup_{p\in\mathbf{R}^n,\alpha\in\mathbf{R}}\{\langle p,x\rangle+\alpha \mid \langle p,y\rangle+\alpha\leq f(y)(\forall y\in\mathbf{Z}^n)\} \qquad (x\in\mathbf{R}^n).$$

An integral polyhedral convex function $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ means a polyhedral convex function with integrality, i.e., f is a polyhedral convex function such that $\mathrm{argmin} f[-p]$ is an integral polyhedron for every $p\in\mathbf{R}^n$. A polyhedral M$^\natural$-convex function that can be obtained as the convex extension of an M$^\natural$-convex function on integer points is said to be integral polyhedral M$^\natural$-convex.

[**Proposition 1**] (Supplement 4.42 in [4], Note 6.54 in [5]) The integral polyhedral M$^\natural$-convexity of f implies that we can take $\alpha_0=1$ in (M$^\natural$-EXC[$\mathbf{R}$]).

A function $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ is said to be convex if its epigraph $\{(x,\alpha)\in\mathbf{R}^n\times\mathbf{R}\mid\alpha\geq f(x)\}$ is a convex set. A convex function with $f>-\infty$ is said to be proper if $\mathrm{dom}_{\mathbf{R}} f\neq\emptyset$, and closed if its epigraph is a closed set. A function $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ is said to be closed proper M$^\natural$-convex if it is closed proper convex, in addition to satisfying (M$^\natural$-EXC[$\mathbf{R}$]).

3. Hybrid M$^\natural$-convex function

Based on the above definitions, we define a hybrid M$^\natural$-convex function.

[**Definition 1**] (hybrid M$^\natural$-convex function) A function $f:\mathbf{Z}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ is said to be hybrid M$^\natural$-convex if it satisfies the following:

There exist a closed proper M$^\natural$-convex function $\hat{f}:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$, a nonsingular matrix $A\in\mathbf{R}^{m\times m}$, and a vector $b\in$

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$\mathbf{R}^m$ such that $\hat{f}(x,y)=f(x,Ay+b)\quad(\forall x\in\mathbf{Z}^n,\forall y\in\mathbf{R}^m)$.

[**Discussion 1**] In the case of $m=0$, a hybrid M$^\natural$-convex function does not necessarily satisfy (M$^\natural$-EXC[$\mathbf{Z}$]) but (M$^\natural$-EXC[$\mathbf{R}$]). On the other hand, a discrete M$^\natural$-convex function satisfies the condition in **Definition 1** with $m=0$. This means that the class of hybrid M$^\natural$-convex functions with $m=0$ is wider than the class of discrete M$^\natural$-convex functions. It is noted that the class of hybrid L$^\natural$-convex functions with $m=0$ coincides with the class of discrete L$^\natural$-convex functions.

For positive integer α and a discrete function $f:\mathbf{Z}^n\to\mathbf{R}\cup\{+\infty\}$, we define a function $f^\alpha:\mathbf{Z}^n\to\mathbf{R}\cup\{+\infty\}$ by

$$f^\alpha(x)=f(\alpha x)\quad(x\in\mathbf{Z}^n).$$

f^α is called a domain scaling of f by α. M$^\natural$-convexity is not preserved under domain scaling, i.e., for an M$^\natural$-convex function f, f^α is not necessarily M$^\natural$-convex [4,5]. This implies the following proposition which is a concrete example of **Discussion 1**.

[**Proposition 2**] A domain scaling of a discrete M$^\natural$-convex function is a hybrid M$^\natural$-convex function with $m=0$.

(Proof.) Let f_1 be a discrete M$^\natural$-convex function. We define

$$f(x)=f_1(\alpha x)\quad(x\in\mathbf{Z}^n).$$

We also let $\hat{f}_1$ be the convex closure of f_1 and

$$\hat{f}(x)=\hat{f}_1(\alpha x)\quad(x\in\mathbf{R}^n).$$

Both $\hat{f}_1$ and $\hat{f}$ are continuous M$^\natural$-convex functions. And we have

$$\hat{f}(x)=\hat{f}_1(\alpha x)=f_1(\alpha x)=f(x)\quad(x\in\mathbf{Z}^n),$$

where the second equality is from Theorem 6.43 (2) in [5] (or Supplement 4.32 in [4]). □

It is noted that the condition in **Proposition 2** is not a necessary and sufficient condition, i.e., not every hybrid M$^\natural$-convex function is given by a domain scaling.

We consider integral polyhedral convexity for a hybrid M$^\natural$-convex function.

[**Definition 2**] (integral polyhedral hybrid M$^\natural$-convex function) A function $f:\mathbf{Z}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ is said to be integral polyhedral hybrid M$^\natural$-convex if $\hat{f}$ is integral polyhedral M$^\natural$-convex in **Definition 1**.

For an integral polyhedral hybrid M$^\natural$-convex function f with $m=0$, we can take $\alpha_0=1$ in (M$^\natural$-EXC[$\mathbf{R}$]) from **Proposition 1**, i.e., we see (M$^\natural$-EXC[$\mathbf{Z}$]). This means that an integral polyhedral hybrid M$^\natural$-convex function f with $m=0$ is discrete M$^\natural$-convex.

In the case where a function $f(x,y)$ is quadratic, in particular, we have the following sufficient condition for a hybrid M$^\natural$-convex function.

[**Proposition 3**] (hybrid quadratic M$^\natural$-convex function) A quadratic function

$$f(x,y)=\begin{bmatrix}x\\y\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}M_d & M_h\\ M_h^{\mathrm{T}} & M_c\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+p^{\mathrm{T}}x+q^{\mathrm{T}}y+r \tag{1}$$

is hybrid quadratic M$^\natural$-convex if there exists a nonsingular matrix $A\in\mathbf{R}^{m\times m}$ such that the matrix

$$M=\begin{bmatrix}M_d & M_hA\\ A^{\mathrm{T}}M_h^{\mathrm{T}} & A^{\mathrm{T}}M_cA\end{bmatrix}=(m_{ij}\mid i,j=1,\cdots,n+m)$$

satisfies the following condition:

$$w^{\mathrm{T}}m_i\geq\min\{0,\min_{j\in\mathrm{supp}^-(w)}w^{\mathrm{T}}m_j\}\quad(\forall w\in\mathbf{R}^n\times\mathbf{R}^m,\forall i\in\mathrm{supp}^+(w)), \tag{2}$$

where m_i denotes the i-th column vector of M.

(Proof.) From Theorem 3.10 in [6], if M satisfies eq. (2),

$$\hat{f}(x,y)=\begin{bmatrix}x\\y\end{bmatrix}^{\mathrm{T}}M\begin{bmatrix}x\\y\end{bmatrix}\quad(x\in\mathbf{R}^n,y\in\mathbf{R}^m) \tag{3}$$

is M$^\natural$-convex. By rewriting eq. (3) we see

$$\begin{aligned}&\hat{f}(x,y)\\&=\begin{bmatrix}x\\Ay+b\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}M_d & M_h\\ M_h^{\mathrm{T}} & M_c\end{bmatrix}\begin{bmatrix}x\\Ay+b\end{bmatrix}+\tilde{p}^{\mathrm{T}}x+\tilde{q}^{\mathrm{T}}y+\tilde{r}\\&=f(x,Ay+b),\end{aligned}$$

which shows that f is hybrid M$^\natural$-convex. □

[**Discussion 2**] The condition of eq. (2) in **Proposition 3** is not a necessary and sufficient condition. For hybrid M$^\natural$-convex function f, it is obvious that there exists a closed proper M$^\natural$-convex function $\hat{f}(x,y)$ but it does not necessarily hold that $\hat{f}(x,y)=f(x,Ay+b)\quad(x\in\mathbf{R}^n,y\in\mathbf{R}^n)$. Here, if a closed proper M$^\natural$-convex function $\hat{f}$ can be written as

$$\hat{f}(x,y)=\text{(the right-hand side of eq. (1) where } Ay+b \text{ is substituted for } y)\quad(x\in\mathbf{R}^n,y\in\mathbf{R}^m), \tag{4}$$

the coefficient matrix of $\hat{f}$ satisfies eq. (2) from Theorem 3.10 in [6].

[**Discussion 3**] Corresponding to **Proposition 3**, a necessary and sufficient condition for a hybrid quadratic L$^\natural$-convex function is given in Proposition 1 in [8]. From **Discussion 2**, we see that the proof for the necessity uses implicitly the fact that a restriction of a continuous L$^\natural$-convex function to integer lattice points is discrete L$^\natural$-convex, which is true of L$^\natural$-convex functions. Since M$^\natural$-convex functions do not have this property, we do not obtain the necessity in **Proposition 3**.

[**Discussion 4**] Referring to **Discussion 1**, we discuss the case of a quadratic function here. Consider eq. (1) with $m=0$, i.e., $f(x)=x^{\mathrm{T}}M_dx+p^{\mathrm{T}}x+r$. The coefficient matrix M_d of f satisfies eq. (2) which is nothing but a characterization of M$^\natural$-convexity for a continuous quadratic function. On the other hand, $l(x)=\dfrac{1}{2}x^{\mathrm{T}}Qx\quad(x\in\mathbf{Z}^n)$ is discrete M$^\natural$-convex if and only if the coefficient matrix Q satisfies

$$q_{ij}\geq 0\quad(\forall i,j\in\{1,\ldots,n\}) \tag{5}$$

$$q_{ij}\geq\min\{q_{ik},q_{jk}\}\ (\forall i,j,k\in\{1,\ldots,n\},k\notin\{i,j\}) \tag{6}$$

[3,6]. It is known that the class of matrices satisfying eq. (2) is wider than the class of matrices satisfying eq. (5) and eq. (6) [6]. When we choose a matrix which satisfies the former and not the latter as the coefficient matrix M_d and put $\hat{f}(x)=f(x)$, then f is hybrid M$^\natural$-convex and not discrete M$^\natural$-convex.

4. Minimization problem and optimality criterion

[**Problem 1**] (hybrid convex optimization problem) Minimize $f(x,y)$ subject to $x\in\mathbf{Z}^n, y\in\mathbf{R}^m$.

Although **Problem 1** is a minimization problem over both of x and y, this problem, by minimizing over each of them successively, can be rewritten as the following equivalent expression.

[**Problem 2**] Minimize $h(x)$ subject to $x\in\mathbf{Z}^n$, where

$$h(x)=\inf_{y\in\mathbf{R}^m} f(x,y). \tag{7}$$

The following proposition describes a property of the function $h(x)$.

[**Proposition 4**] For an integral polyhedral hybrid M$^\natural$-convex function $f(x,y)$, the function $h(x)$ is discrete M$^\natural$-convex.

(Proof.) For f, there exists an integral polyhedral M$^\natural$-convex function $\hat{f}$ such that

$$\hat{f}(x,y)=f(x,Ay+b) \quad (x\in\mathbf{Z}^n).$$

The projection of an integral polyhedral M$^\natural$-convex function $\hat{f}$, defined by

$$\hat{h}(x)=\inf_{y\in\mathbf{R}^m}\hat{f}(x,y) \quad (x\in\mathbf{R}^n),$$

is integral polyhedral M$^\natural$-convex. Here, for $x\in\mathbf{Z}^n$, we have

$$\begin{aligned} h(x) &= \inf_{y\in\mathbf{R}^m} f(x,y)=\inf_{y\in\mathbf{R}^m} f(x,Ay+b)\\ &= \inf_{y\in\mathbf{R}^m}\hat{f}(x,y)=\hat{h}(x) \quad (x\in\mathbf{R}^n), \end{aligned}$$

which means that h can be extended to an integral polyhedral M$^\natural$-convex function. Hence, h is discrete M$^\natural$-convex. □

We consider, as in **Discussion 2**, a quadratic function f of eq. (1) with $\hat{f}$ described by eq. (4). Under the assumption that M_c is a nonsingular matrix, h is described by

$$\begin{aligned} h(x)=x^{\mathrm{T}}(M_d-M_hM_c^{-1}M_h^{\mathrm{T}})x+(p-M_hM_c^{-1}q)^{\mathrm{T}}x\\ +r-\frac{1}{4}q^{\mathrm{T}}M_c^{-1}q. \end{aligned}$$

We now show an optimality criterion for integral polyhedral hybrid M$^\natural$-convex function minimization.

[**Lemma 1**] Suppose that $f(x,y)$ is hybrid convex. If $h(x)$ defined by eq. (7) is M$^\natural$-convex, then, the following equivalence holds for $(x^*,y^*)\in\mathbf{Z}^n\times\mathbf{R}^m$:

$$\begin{aligned} & f(x^*,y^*)\le f(x,y) \quad (\forall(x,y)\in\mathbf{Z}^n\times\mathbf{R}^m)\\ \Longleftrightarrow & \begin{cases} 0\in\partial_y f(x^*,y^*)\\ h(x^*)\le h(x^*-\chi_u+\chi_v) \quad (\forall u,v\in V)\\ h(x^*)\le h(x^*\pm\chi_v) \quad (\forall v\in V), \end{cases} \end{aligned} \tag{8}$$

where $\partial_y f(x,y)$ is the subdifferential of function $f(x,y)$ at (x,y) with respect to y, i.e., $\partial_y f(x,y)=\{p\in\mathbf{R}^m \mid f(x,y')-f(x,y)\ge\langle p,y'-y\rangle \quad (\forall y'\in\mathbf{R}^m)\}$, see (2.58) in [2], (2.24) in [4], and (3.23) in [5].

(Proof.) It is trivial to show [⇒]. We prove [⇐]. From the first equation in eq. (8) and Theorems 3.22 and 3.23 in [2], we have $f(x^*,y^*)\le f(x^*,y)$ for any $y\in\mathbf{R}^m$, and hence $h(x^*)=f(x^*,y^*)$. Combining M$^\natural$-convexity of $h(x)$, the second equation in eq. (8), and Theorem 6.26 (2) in [5] (or Theorem 4.16 (2) in [4]), we obtain $h(x^*)\le h(x)$ for any $x\in\mathbf{Z}^n$. Putting all together, we have

$$f(x^*,y^*)=h(x^*)\le h(x)\le f(x,y)$$

for all $x\in\mathbf{Z}^n$ and $y\in\mathbf{R}^m$. □

[**Theorem 1**] (optimality criterion) Let $f(x,y)$ be integral polyhedral hybrid M$^\natural$-convex and $h(x)$ be defined by eq. (7). Then, for $(x^*,y^*)\in\mathbf{Z}^n\times\mathbf{R}^m$ we have the following equivalence:
$f(x^*,y^*)\le f(x,y) \quad (\forall(x,y)\in\mathbf{Z}^n\times\mathbf{R}^m) \iff$ eq. (8).

(Proof.) Because $f(x,y)$ is integral polyhedral hybrid M$^\natural$-convex, from **Proposition 4**, $h(x)$ is M$^\natural$-convex. Now, we can use **Lemma 1** to show the equivalence. □

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