研 究 速 報

English translation: 連続/離散ハイブリッド凸最適化とその最適性規準*

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Continuous/Discrete Hybrid Convex Optimization and Its Optimality Criterion *

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1. Introduction

The algorithm which solves a mixed integer quadratic problem (MIQP) quickly is required recently in the filed of model predictive control for hybrid systems [1]. Although an MIQP, generally, is solved by branch and bound method, it takes exponential time in the worst case [6]. If we provide a class of problems tractable to solve with polynomial time algorithms, it might be profitable in practice. This paper, as the first step, proposes a continuous/discrete hybrid convex optimization problem and shows its optimality condition. We start with the definition of a hybrid convex function.

[**Definition 1**] (hybrid convex function) A function $f:\mathbf{Z}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ is said to be hybrid convex if if it is convex extensible.

Here, a function $f:\mathbf{Z}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ is said to be convex extensible if there exists a convex function $g:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ such that $g(x,y)=f(x,y)$ $(\forall x\in\mathbf{Z}^n,\forall y\in\mathbf{R}^m)$. We show an example of a quadratic function.

[**Example 1**] (hybrid convex function) Suppose that $x=(x_1,x_2)\in\mathbf{Z}^2$ and $y\in\mathbf{R}$. A quadratic function

$$f(x,y)=\left[\begin{array}{c}x_1-p\\ x_2-q\\ \hline y-r\end{array}\right]^{\mathrm{T}}\left[\begin{array}{cc|c}9&-6&1\\ -6&8&0\\ \hline 1&0&0.5\end{array}\right]\left[\begin{array}{c}x_1-p\\ x_2-q\\ \hline y-r\end{array}\right] \quad (1)$$

is hybrid convex. Here, $p,q,r\in\mathbf{R}$ are parameters.

We describe an optimization problem for an hybrid convex function defined by **Definition 1** as follows:

[**Problem 1**] (hybrid optimization problem)

$$\text{Minimize } f(x,y) \quad \text{subject to } x\in\mathbf{Z}^n, y\in\mathbf{R}^m.$$

We deal with this type of optimization problems from the next section. Based on $\mathrm{L}^\natural$-convex functions from recent research in discrete convex analysis, we first clarify a class of tractable hybrid convex functions. To be specific, we define a hybrid $\mathrm{L}^\natural$-convex function in Section **2.**, and show an optimality condition for its minimization in Section **3.**

2. Hybrid $\mathrm{L}^\natural$-convex function

[**Definition 2**] (hybrid $\mathrm{L}^\natural$-convex function) A function $f:\mathbf{Z}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$ is said to be hybrid $\mathrm{L}^\natural$-convex if it satisfies the following:

> There exist an $\mathrm{L}^\natural$-convex function $g:\mathbf{R}^n\times\mathbf{R}^m\to\mathbf{R}\cup\{+\infty\}$, a nonsingular matrix $A\in\mathbf{R}^{m\times m}$, and a vector $b\in\mathbf{R}^m$ such that $g(x,y)=f(x,Ay+b)$ $(\forall x\in\mathbf{Z}^n,\forall y\in\mathbf{R}^m)$.

An $\mathrm{L}^\natural$-convex function is defined as follows.

[**Definition 3**] ($\mathrm{L}^\natural$-convex function[5]) [1] We call a function $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ $\mathrm{L}^\natural$-convex if f satisfies translation submodularity **($\mathbf{SBF}^\natural$)**.

$$(\mathbf{SBF}^\natural)\quad f(x)+f(x')\geq f((x-\alpha\mathbf{1})\vee x') +f(x\wedge(x'+\alpha\mathbf{1})) \quad (\forall x,x'\in\mathbf{R}^n,\forall\alpha\geq 0),$$

where we denote $\mathbf{1}=(1,\cdots,1)^{\mathrm{T}}$.

In the case where a function $f(x,y)$ is quadratic, in particular, a hybrid $\mathrm{L}^\natural$-convex function is described as follows.

[**Proposition 1**] (hybrid quadratic $\mathrm{L}^\natural$-convex function) A quadratic function

$$f(x,y)=\left[\begin{array}{c}x\\ y\end{array}\right]^{\mathrm{T}}\left[\begin{array}{cc}L_d&L_h\\ L_h^{\mathrm{T}}&L_c\end{array}\right]\left[\begin{array}{c}x\\ y\end{array}\right]+p^{\mathrm{T}}x+q^{\mathrm{T}}y+r \quad (2)$$

is hybrid quadratic $\mathrm{L}^\natural$-convex if and only if there exists a nonsingular matrix $A\in\mathbf{R}^{m\times m}$ such that matrix

$$L=\left[\begin{array}{cc}L_d&L_hA\\ A^{\mathrm{T}}L_h^{\mathrm{T}}&A^{\mathrm{T}}L_cA\end{array}\right]=(\ell_{ij}\,|\,i,j=1,\cdots,n+m)$$

satisfies the following conditions:

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[1] To be more precise, the function must be assumed to be a closed convex function.

$$\ell_{ij} \le 0 \qquad (i \neq j; 1 \le i,j \le n+m), \tag{3}$$

$$\sum_{j=1}^{n+m} \ell_{ij} \ge 0 \qquad (1 \le i \le n+m). \tag{4}$$

(Proof.) [Sufficiency] In **Definition 2**, we put $g(x,y) =$ (the right-hand side of eq. (2) where $Ay+b$ is substituted for y) $(x \in \mathbf{R}^n, y \in \mathbf{R}^m)$. From Theorem 2.24 in [3], g satisfies **(SBF$^\natural$)**.

[Necessity] We can prove this in the same way as Theorem 2.24 in [3]. □

The quadratic function in eq. (1) is hybrid L$^\natural$-convex. If we choose $A = -2$ as the nonsingular matrix A, the matrix L is calculated as

$$L = \left[\begin{array}{cc|c} 9 & -6 & -2 \\ -6 & 8 & 0 \\ \hline -2 & 0 & 2 \end{array}\right].$$

This satisfies conditions in eq. (3) and eq. (4).

Although **Problem 1** is a minimization problem over both of x and y, this problem, by minimizing over each of them successively, can be rewritten as the following equivalent expression.

[**Problem 2**]

Minimize $h(x)$ subject to $x \in \mathbf{Z}^n$,

where

$$h(x) = \inf_{y \in \mathbf{R}^m} f(x,y). \tag{5}$$

The following proposition describes a property of the function $h(x)$.

[**Proposition 2**] For a hybrid L$^\natural$-convex function $f(x,y)$, the function $h(x)$ is discrete L$^\natural$-convex.

(Proof.) We prove that h has translational submodularity **(SBF$^\natural$)**. From the definition of h, for all $\varepsilon > 0$, there exists $y, y' \in \mathbf{R}^m$ such that $h(x) + \varepsilon \ge f(x,y)$, $h(x') + \varepsilon \ge f(x',y')$. Adding these two inequalities, we have

$$h(x) + h(x') + 2\varepsilon \ge f(x,y) + f(x',y').$$

The translation-submodularity of f and the definition of h yield

$$\begin{aligned} f(x,y) + f(x',y') &\ge f((x-\alpha\mathbf{1}) \vee x', (y-\alpha\mathbf{1}) \vee y') \\ &\quad + f((x \wedge (x'+\alpha\mathbf{1}), y \wedge (y'+\alpha\mathbf{1})) \\ &\ge h((x-\alpha\mathbf{1}) \vee x') + h(x \wedge (x'+\alpha\mathbf{1})). \end{aligned}$$

Since we may take $\varepsilon > 0$ arbitrarily small, we have

$$h(x) + h(x') \ge h((x-\alpha\mathbf{1}) \vee x') + h(x \wedge (x'+\alpha\mathbf{1})).$$

This means that h is L$^\natural$-convex over $\mathbf{Z}^n$. □

We apply **Proposition 2** to an quadratic function. We consider the quadratic function in eq. (2). Suppose that L_c is nonsingular. Then

$$h(x) = x^{\mathrm{T}}(L_d - L_h L_c^{-1} L_h^{\mathrm{T}})\ x + (p - L_h L_c^{-1} q)^{\mathrm{T}} x + r - \frac{1}{4} q^{\mathrm{T}} L_c^{-1} q.$$

This means that h is quadratic L$^\natural$-convex if and only if the matrix $L_d - L_h L_c^{-1} L_h^{\mathrm{T}}$ satisfies eq. (3) and eq. (4). A known fact in linear algebra, "if a matrix L satisfies eq. (3) and (4), then $L_d - L_h L_c^{-1} L_h^{\mathrm{T}}$ also satisfies eq. (3) and eq. (4)," corresponds to **Proposition 2**.

Finally, we show the following example where f is not hybrid L$^\natural$-convex but h is L$^\natural$-convex.

[**Example 2**] Function

$$f(x_1,x_2,y_1,y_2) = \begin{bmatrix} x_1 \\ x_2 \\ \hline y_1 \\ y_2 \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{cc|cc} 3 & 1 & 1 & -1 \\ 1 & 6 & 2 & 1 \\ \hline 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline y_1 \\ y_2 \end{bmatrix}$$

is not hybrid L$^\natural$-convex. Indeed, there is no nonsingular matrix A with which L satisfies eq. (3) because the (1,2) entry is positive. On the other hand, $L_d - L_h L_c^{-1} L_h^{\mathrm{T}}$ is an identity matrix and satisfies eq. (3) and eq. (4), and hence $h(x_1,x_2)$ is L$^\natural$-convex.

3. Optimality criterion

This section shows an optimality criterion for a hybrid L$^\natural$-convex function defined in **2.**

[**Lemma 1**] Suppose that $f(x,y)$ is hybrid convex. If $h(x)$ defined by eq. (5) is L$^\natural$-convex, then, the following equivalence holds for $(x^*,y^*) \in \mathbf{Z}^n \times \mathbf{R}^m$:

$$\begin{aligned} &f(x^*,y^*) \le f(x,y) \quad (\forall (x,y) \in \mathbf{Z}^n \times \mathbf{R}^m) \\ &\Longleftrightarrow \begin{cases} 0 \in \partial_y f(x^*,y^*) \\ h(x^*) \le h(x^* \pm e) \quad (\forall e \in \{0,1\}^n), \end{cases} \end{aligned} \tag{6}$$

where $\partial_y f(x,y)$ is the subdifferential of function $f(x,y)$ at (x,y) with respect to y ((2.58) in [2] and (2.24) in [3]), and $e \in \{0,1\}^n$ is any n-dimensional $\{0,1\}$ vector.

(Proof.) It is trivial to show [⇒]. We prove [⇐].

From the first equation in eq. (6) and Theorems 3.22 and 3.23 in [2], we have $f(x^*,y^*) \le f(x^*,y)$ for any $y \in \mathbf{R}^m$, and hence $h(x^*) = f(x^*,y^*)$. Combining L$^\natural$-convexity of $h(x)$, the second equation in eq. (6), and Theorem 5.12 (2) in [3], we obtain $h(x^*) \le h(x)$ for any $x \in \mathbf{Z}^n$. Putting all together, we have

$$f(x^*,y^*) = h(x^*) \le h(x) \le f(x,y)$$

for all $x \in \mathbf{Z}^n$ and $y \in \mathbf{R}^m$. □

[**Theorem 1**] (optimality criterion) Let $f(x,y)$ be hybrid L$^\natural$-convex and $h(x)$ be defined by eq. (5). Then, for $(x^*,y^*) \in \mathbf{Z}^n \times \mathbf{R}^m$ we have the following equivalence:

$$\begin{aligned} &f(x^*,y^*) \le f(x,y) \quad (\forall (x,y) \in \mathbf{Z}^n \times \mathbf{R}^m) \\ &\Longleftrightarrow \begin{cases} 0 \in \partial_y f(x^*,y^*) \\ h(x^*) \le h(x^* \pm e) \quad (\forall e \in \{0,1\}^n). \end{cases} \end{aligned} \tag{7}$$

(Proof.) Because $f(x,y)$ is hybrid L$^\natural$-convex, from **Proposition 2**, $h(x)$ is L$^\natural$-convex. Now, we can use **Lemma 1** to show the equivalence. □

Although a straightforward verification of eq. (6) and eq. (7) requires $O(2^n)$ function evaluations, it is known that the verification can be done in polynomial time by submodular function minimization algorithms [4].

We demonstrate this theorem by an example of a quadratic function.

[**Example 3**] We consider a minimization problem of $f(x,y)$ defined in **Example 1**. Since the function f is smooth in y, condition $0 \in \partial_y f$ is rewritten as $\nabla_y f = 0$. From this, two conditions in eq. (7) are described as follows:

$$\nabla_y f(x_1^*,x_2^*,y^*) = 2(x_1^*-p)+(y^*-r) = 0, \tag{8}$$

$$\frac{7}{6}(x_1^*-p)-\frac{7}{12} \le x_2^*-q \le \frac{7}{6}(x_1^*-p)+\frac{7}{12}, \tag{9}$$

$$\frac{3}{4}(x_1^*-p)-\frac{1}{2} \le x_2^*-q \le \frac{3}{4}(x_1^*-p)+\frac{1}{2}, \tag{10}$$

$$-\frac{1}{2}(x_1^*-p)-\frac{3}{4} \le x_2^*-q \le -\frac{1}{2}(x_1^*-p)+\frac{3}{4}. \tag{11}$$

For example, when we take $p=\frac{7}{4}, q=\frac{7}{4}, r=\frac{11}{4}$, the region described by eq. (9), eq. (10), and eq. (11) is shown in Fig.1. The minimizer of this function is $(x_1^*,x_2^*)=(2,2)$ and there is no other integer solution. Furthermore, we have $y^*=\frac{9}{4}$ by eq. (8).

As another example, when we take $p=\frac{3}{2}, q=\frac{3}{2}, r=5$, the region described by eq. (9), eq. (10), and eq. (11) is shown in Fig.2. In this case, there exist two minimizers $(x_1^*,x_2^*)=(1,1)$ and $(2,2)$. There is no other integer solution. Furthermore, we have $(x_1^*,x_2^*,y^*)=(1,2,6)$ and $(2,2,4)$ by eq. (8).

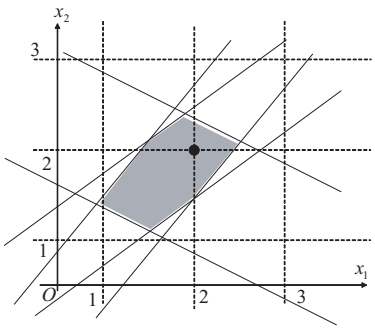

Fig. 1 The region including minimal point for $p = 7/4, q = 7/4$

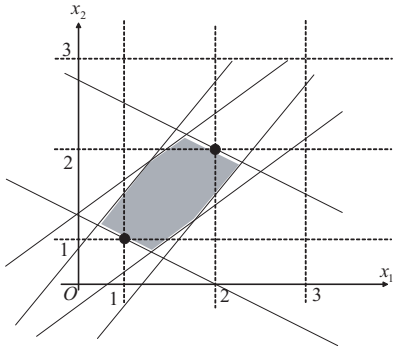

Fig. 2 The region including minimal point for $p = 3/2, q = 3/2$

4. Conclusion

As future work, it would be important (1) to extend this argument to an $M^\natural$-convex function, which is another type of discrete convex functions, and (2) to develop an optimization algorithm for hybrid convex functions based on the optimality criterion given in this paper.

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