Proof of Biconjugacy of Univariate Discrete Convex Functions

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A function $f : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is said to be a *discrete convex function* (or simply *convex function*) if $\mathrm{dom}\, f \neq \emptyset$ and

$$f(x-1) + f(x+1) \geq 2f(x) \qquad (\forall x \in \mathbb{Z}), \tag{1}$$

where it is understood that this inequality is satisfied trivially if $f(x-1) = +\infty$ or $f(x+1) = +\infty$. The inequality (1) can be rewritten as

$$f(x) - f(x-1) \leq f(x+1) - f(x) \qquad (\forall x \in \mathrm{dom}\, f), \tag{2}$$

showing the monotonicity (nondecrease) of the difference $f(x+1) - f(x)$ on $\mathrm{dom}\, f$. Here we use notation

$$\mathrm{dom}\, f = \{x \in \mathbb{Z} \mid -\infty < f(x) < +\infty\} \tag{3}$$

for the effective domain of f.

For an integer-valued function f on $\mathbb{Z}$, we define a transformation of $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ to $f^{\bullet} : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ by

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z}). \tag{4}$$

This function $f^{\bullet}$ is referred to as the *integral conjugate* of f, and the mapping $f \mapsto f^{\bullet}$ of (4) as the *fully-discrete* or *fully-integral* Legendre–Fenchel transformation.

For a function $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$, we can apply the fully-integral Legendre–Fenchel transformation (4) twice to obtain $f^{\bullet\bullet} = (f^{\bullet})^{\bullet}$. That is,

$$f^{\bullet\bullet}(x) = \sup\{px - f^{\bullet}(p) \mid p \in \mathbb{Z}\} \qquad (x \in \mathbb{Z}). \tag{5}$$

The resulting function $f^{\bullet\bullet}$ is referred to as the *integral biconjugate* of f.

The conjugacy theorem for univariate discrete convex functions reads as follows.

Theorem 1. *For an integer-valued univariate discrete convex function $f : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$, the integral conjugate $f^{\bullet}$ in* (4) *is another integer-valued univariate discrete convex function. Furthermore, the integral biconjugate $f^{\bullet\bullet}$ in* (5) *coincides with f itself, i.e., $f^{\bullet\bullet} = f$.*

Proof. The proof consists of three parts.

[Discrete convexity (1) of $f^{\bullet}$]: The addition of the two expressions

$$f^{\bullet}(p-1) = \sup_x\{(p-1)x - f(x)\}, \quad f^{\bullet}(p+1) = \sup_x\{(p+1)x - f(x)\}$$

yields

$$\begin{aligned} & f^{\bullet}(p-1) + f^{\bullet}(p+1) \\ & \geq \sup_x\{((p+1)x - f(x)) + ((p-1)x - f(x))\} \\ & = 2\sup_x\{px - f(x)\} = 2f^{\bullet}(p). \end{aligned}$$

This shows the discrete convexity (1) of $f^{\bullet}$.

$[f^{\bullet\bullet} \leq f]$: For any $x, p \in \mathbb{Z}$ we have $f^{\bullet}(p) \geq px - f(x)$ by (4), and hence $px - f^{\bullet}(p) \leq f(x)$. Therefore, $f^{\bullet\bullet}(x) = \sup_p \{px - f^{\bullet}(p)\} \leq f(x)$.

$[f^{\bullet\bullet} \geq f]$: First we assume $x \in \mathrm{dom}\, f$. Take an integer p satisfying

$$f(x) - f(x-1) \leq p \leq f(x+1) - f(x), \tag{6}$$

which is possible by (2) and the integrality of the function value. Consider the function $h(y) = f(y) - py$ in $y \in \mathbb{Z}$. Then we have

$$h(x) \leq \min\{h(x-1), h(x+1)\}$$

by (6). This implies that $h(x) \leq h(y)$ for all $y \in \mathbb{Z}$, which is equivalent to $px - f(x) \geq py - f(y)$ $(y \in \mathbb{Z})$. Hence we have

$$px - f(x) = \sup_y \{py - f(y)\} = f^{\bullet}(p).$$

Therefore,

$$f^{\bullet\bullet}(x) = \sup_q \{qx - f^{\bullet}(q)\} \geq px - f^{\bullet}(p) = f(x).$$

Next we consider the case of $x \notin \mathrm{dom}\, f$. We assume that $\mathrm{dom}\, f$ is an integer interval $[a, b]_{\mathbb{Z}}$ with b finite and $x \geq b + 1$; the other case with a finite and $x \leq a - 1$ can be treated similarly. For all sufficiently large $p \in \mathbb{Z}$, say $p \geq p_0$, we have $f^{\bullet}(p) = pb - f(b)$. Therefore,

$$f^{\bullet\bullet}(x) = \sup_{p \in \mathbb{Z}} \{px - f^{\bullet}(p)\} \geq \sup_{p \geq p_0} \{px - f^{\bullet}(p)\} = \sup_{p \geq p_0} \{p(x - b) + f(b)\} = +\infty.$$

Therefore, $f^{\bullet\bullet}(x) = +\infty = f(x)$. □

Theorem 1 above shows that the fully-integral Legendre–Fenchel transformation (4) establishes a symmetric (or involutive) one-to-one correspondence within the class of integer-valued univariate discrete convex functions defined on $\mathbb{Z}$.

Example 1. Consider a quadratic function $f(x) = x^2$ in a continuous or discrete variable x. The (ordinary) conjugate function in the continuous case ($x \in \mathbb{R}$) is given by

$$f^{\bullet}(p) = \sup\{px - x^2 \mid x \in \mathbb{R}\} = \frac{1}{4}p^2 \qquad (p \in \mathbb{R}).$$

In contrast, the conjugate function (4) in the discrete case ($x \in \mathbb{Z}$) is given by

$$\begin{aligned} f^{\bullet}(p) &= \sup\{px - x^2 \mid x \in \mathbb{Z}\} \\ &= \max\{px - x^2 \mid x \in \{\lfloor p/2 \rfloor, \lceil p/2 \rceil\}\} \\ &= \max\left\{ \left\lfloor \frac{p}{2} \right\rfloor (p - \left\lfloor \frac{p}{2} \right\rfloor), \left\lceil \frac{p}{2} \right\rceil (p - \left\lceil \frac{p}{2} \right\rceil) \right\} \\ &= \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil \qquad (p \in \mathbb{Z}). \end{aligned} \tag{7}$$

This is a piecewise-linear convex function whose graph consists of line segments connecting $(2k-1, k(k-1))$ and $(2k+1, k(k+1))$ for $k \in \mathbb{Z}$. Furthermore, for $x \in \mathbb{Z}$ we have

$$f^{\bullet\bullet}(x) = \sup\left\{ px - \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil \mid p \in \mathbb{Z} \right\} = x^2 = f(x) \qquad (x \in \mathbb{Z}),$$

where the supremum is attained by $p = 2\lfloor |x|/2 \rfloor + 1$. Thus we obtain the integral biconjugacy $f^{\bullet\bullet} = f$, as stated in Theorem 1. ■

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