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Errata and Supplements:
"M-Convex Function on Generalized Polymatroid"

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- Page 96, two lines from bottom (inequality in (M$^\natural$-EXC$_{\rm W}$)): The inequality is intended to mean

$$f(x) + f(y) \geq \min \left[\min_{u \in \mathrm{supp}^+(x-y)} \left\{ f(x - \chi_u) + f(y + \chi_u) \right\}, \right. \\ \left. \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} \left\{ f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \right\} \right].$$

- Page 99, the first paragraph in Section 4: the second and the third "$\forall$" symbols in (M-EXC$_{\rm W}$) should be "$\exists$." That is, the condition (M-EXC$_{\rm W}$) should be as follows:

  (M-EXC$_{\mathbf{W}}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that $f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$.

- Page 99, two lines from bottom (inequality in (M$^\natural$-EXC$_{\rm pW}$)): The inequality is intended to mean

$$f(x) + f(y) \geq \min \left[\min_{u \in \mathrm{supp}^+(x-y)} \left\{ f(x - \chi_u) + f(y + \chi_u) \right\}, \right. \\ \left. \min_{\substack{u \in \mathrm{supp}^+(x-y) \\ v \in \mathrm{supp}^-(x-y)}} \left\{ f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \right\} \right].$$

- Page 100, Lemma 4.3 : The omitted proof is given:
  Assuming (M$^\natural$-EXC$_{\rm W}$), we show that for any $x, y \in \mathrm{dom}\, f$ with $x(V) < y(V)$,

$$f(x) + f(y) \geq \min_{v \in \mathrm{supp}^-(x-y)} \{ f(x + \chi_v) + f(y - \chi_v) \}.$$

  Set $\mathcal{D} = \{(x, y) \mid x, y \in \mathrm{dom}\, f,\ x(V) < y(V),\ \forall v \in \mathrm{supp}^-(x - y) :\ f(x) + f(y) < f(x + \chi_v) + f(y - \chi_v)\}$. We assume $\mathcal{D} \neq \emptyset$ and derive a contradiction.

  Let (x, y) be the element in $\mathcal{D}$ which minimizes the value $\|x - y\|_1$. Using $\varepsilon (> 0)$, we set $p \in \mathbf{R}^V$ as follows:

$$p(v) = \begin{cases} f(x) - f(x + \chi_v) & (v \in \mathrm{supp}^-(x - y),\ x + \chi_v \in \mathrm{dom}\, f), \\ f(y - \chi_v) - f(y) - \varepsilon & (v \in \mathrm{supp}^-(x - y),\ x + \chi_v \notin \mathrm{dom}\, f,\ y - \chi_v \in \mathrm{dom}\, f), \\ 0 & (\text{otherwise}). \end{cases}$$

Define $f_p(x) = f(x) + \sum\{p(w)x(w) \mid w \in V\}$ $(\forall x \in \mathbf{Z}^V)$.

Claim 1:

$$\begin{aligned} f_p(x+\chi_v) &= f_p(x) \qquad (v \in \mathrm{supp}^-(x-y),\ x+\chi_v \in \mathrm{dom}\, f), && (1)\\ f_p(y-\chi_v) &> f_p(y) \qquad (v \in \mathrm{supp}^-(x-y)). && (2) \end{aligned}$$

Claim 2: $\exists u_1 \in \mathrm{supp}^+(x-y)$, $\exists v_1 \in \mathrm{supp}^-(x-y)$ such that $y+\chi_{u_1}-\chi_{v_1} \in \mathrm{dom}\, f$.

(Proof of Claim 2) Since $y(V) > x(V)$, we can apply (M$^\natural$-EXC$_\mathrm{W}$) to y, x. Combining with the fact $(x,y) \in \mathcal{D}$, there exist $u_1 \in \mathrm{supp}^+(x-y)$, $v_1 \in \mathrm{supp}^-(x-y)$ such that

$$(+\infty >) f(x) + f(y) \geq f(x-\chi_{u_1}+\chi_{v_1}) + f(y+\chi_{u_1}-\chi_{v_1}).$$

Hence $y+\chi_{u_1}-\chi_{v_1} \in \mathrm{dom}\, f$. (end of proof for Claim 2)

Suppose that $u_1 \in \mathrm{supp}^+(x-y)$, $v_1 \in \mathrm{supp}^-(x-y)$ satisfy

$$f_p(y+\chi_{u_1}-\chi_{v_1}) = \min_{\substack{u \in \mathrm{supp}^+(x-y)\\ v \in \mathrm{supp}^-(x-y)}} f_p(y+\chi_u-\chi_v). \qquad (3)$$

The previous claim yields that $f_p(y+\chi_{u_1}-\chi_{v_1}) < +\infty$. Put $y' = y+\chi_{u_1}-\chi_{v_1}$.

Claim 3: $(x,y') \in \mathcal{D}$.

(Proof of Claim 3) Since $y'(V) = y(V) > x(V)$, we have only to show that

$$f_p(x) + f_p(y') < f_p(x+\chi_v) + f_p(y'-\chi_v) \qquad (4)$$

for each $v \in \mathrm{supp}^-(x-y')$. This inequality holds obviously when $x+\chi_v \notin \mathrm{dom}\, f$. Therefore we assume that $x+\chi_v \in \mathrm{dom}\, f$. We have $f_p(x) = f_p(x+\chi_v)$ by (1), and

$$\begin{aligned} & f_p(y'-\chi_v)\\ &= f_p(y) + f_p(y+\chi_{u_1}-\chi_{v_1}-\chi_v) - f_p(y)\\ &\geq \min\{f_p(y+\chi_{u_1}-\chi_{v_1}) + f_p(y-\chi_v),\\ &\qquad f_p(y+\chi_{u_1}-\chi_v) + f_p(y-\chi_{v_1})\} - f_p(y) \qquad (\text{by (M}^\natural\text{-EXC}_\mathrm{W}))\\ &\geq \min\{f_p(y-\chi_v) - f_p(y), f_p(y-\chi_{v_1}) - f_p(y)\} + f_p(y+\chi_{u_1}-\chi_{v_1}) \qquad (\text{by (3)})\\ &> f_p(y') \qquad (\text{by (2)}). \end{aligned}$$

Thus, the inequality (4) holds. (end of proof for Claim 3)

We have $(x,y') \in \mathcal{D}$ by Claim 3, which contradicts the choice of (x,y) since $\|x - y'\|_1 = \|x-y\|_1 - 2$. Therefore $\mathcal{D} = \emptyset$.

(end)

- Page 102, Remark 5.2 : A direct proof for the equivalence between (G-EXC) and (G-EXC$_0$) for $Q \subseteq \mathbf{Z}^V$ is given here.

We first note that with the notation $\chi_0 = \mathbf{0}$, the exchange properties (G-EXC) and (G-EXC$_0$) can be rewritten in a compact form as follows:

(G-EXC) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y) \cup \{0\}$:
$x - \chi_u + \chi_v \in Q$ and $y + \chi_u - \chi_v \in Q$,

(G-EXC$_\mathbf{0}$) $\forall x, y \in Q$, $\forall u \in \mathrm{supp}^+(x-y)$, both (i) and (ii) hold, where
(i) $\exists v \in \mathrm{supp}^-(x-y) \cup \{0\}$: $x - \chi_u + \chi_v \in Q$,
(ii) $\exists w \in \mathrm{supp}^-(x-y) \cup \{0\}$: $y + \chi_u - \chi_w \in Q$.

The implication "(G-EXC) $\Longrightarrow$ (G-EXC$_0$)" is obvious. To prove the converse, let $Q \subseteq \mathbf{Z}^V$ be a nonempty set satisfying (G-EXC$_0$) and show that Q satisfies (G-EXC).

Claim 1: (G-EXC) holds for $x, y \in Q$ with $x \leq y$ or $x \geq y$.
[Proof of Claim 1] If $x \leq y$ then (G-EXC) holds trivially since $\mathrm{supp}^+(x-y) = \emptyset$. Suppose $x \geq y$ and $x \neq y$. For $u \in \mathrm{supp}^+(x-y)$, (G-EXC$_0$) implies that $x' = x - \chi_u \in Q$ and $y' = y + \chi_u \in Q$ since $\mathrm{supp}^-(x-y) = \emptyset$. Thus, (G-EXC) holds. [End of Claim 1]

We denote

$$\|z\|_1^+ = \sum_{u \in V} \max(0, z(u)), \qquad \|z\|_1^- = \sum_{u \in V} \max(0, -z(u)) \qquad (z \in \mathbf{Z}^V).$$

Claim 2: For every $x, y \in Q$ with $\|x-y\|_1^+ = 2$ and $\|x-y\|_1^- = 1$, there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in Q$ and $y + \chi_u - \chi_v \in Q$.
[Proof of Claim 2] To simplify the notation, we let $x = z + \chi_u + \chi_{u'}$ and $y = z + \chi_v$ for $z \in \mathbf{Z}^V$ and $u, u', v \in V$ with $v \notin \{u, u'\}$. We denote $z(u,v) = z + \chi_u + \chi_v$, $z(v) = z + \chi_v$, $z(u,u',v) = z + \chi_u + \chi_{u'} + \chi_v$, and so on.
By applying (G-EXC$_0$) to $x = z(u,u'), y = z(v)$, and u', we have

$$z(u) \in Q \text{ or } z(u,v) \in Q \text{ (or both)}, \tag{5}$$
$$z(u') \in Q \text{ or } z(u',v) \in Q \text{ (or both)}. \tag{6}$$

By applying (G-EXC$_0$) to $y = z(v), x = z(u,u')$, and v, we have

$$\text{at least one of } z \in Q, z(u) \in Q, \text{ and } z(u') \in Q \text{ holds}, \tag{7}$$
$$\text{at least one of } z(u,u',v) \in Q, z(u',v) \in Q, \text{ and } z(u,v) \in Q \text{ holds}. \tag{8}$$

Based on (5) and (6), we consider the following three cases:

Case 1: $z(u) \in Q$ and $z(u',v) \in Q$ holds, or $z(u') \in Q$ and $z(u,v) \in Q$ holds,
Case 2: $z(u) \notin Q$ and $z(u') \notin Q$,
Case 3: $z(u',v) \notin Q$ and $z(u,v) \notin Q$.

The claim holds in Case 1; below we will derive contradictions in Cases 2 and 3.

We consider Case 2. By (5) and (7), we have $z(u,v) \in Q$ and $z \in Q$. By applying (G-EXC$_0$) to $z(u,v), z$, and u, we have $z(u) \in Q$, a contradiction.

We then consider Case 3. By (6) and (8), we have $z(u') \in Q$ and $z(u,u',v) \in Q$. By applying (G-EXC$_0$) to $z(u,u',v), z(u)$, and v, we have $z(u,v) \in Q$, a contradiction.

[End of Claim 2]

Claim 3: For every $x, y \in Q$ with $\|x-y\|_1 \leq 3$ and every $u \in \mathrm{supp}^+(x-y)$, there exists some $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in Q$ and $y + \chi_u - \chi_v \in Q$.

[Proof of Claim 3] If $x \leq y$ or $x \geq y$, then the claim follows from Claim 1. Hence, we assume $\mathrm{supp}^+(x-y) \neq \emptyset$ and $\mathrm{supp}^-(x-y) \neq \emptyset$. Then, we have

$$(\|x-y\|_1^+, \|x-y\|_1^-) \in \{(2,1),(1,2),(1,1)\}.$$

If $(\|x-y\|_1^+, \|x-y\|_1^-) = (1,1)$, then we have $y = x - \chi_u + \chi_v$ for some $u, v \in V$, and therefore the claim holds. Finally, if $(\|x-y\|_1^+, \|x-y\|_1^-) \in \{(2,1),(1,2)\}$, then the claim follows from Claim 2. [End of Claim 3]

Set

$$\mathcal{D} = \{(x,y) \mid x, y \in Q,\ \exists u_* \in \mathrm{supp}^+(x-y),\ \forall v \in \mathrm{supp}^-(x-y) \cup \{0\} : \\ x - \chi_{u_*} + \chi_v \notin Q \text{ or } y + \chi_{u_*} - \chi_v \notin Q\}.$$

We assume $\mathcal{D} \neq \emptyset$ and derive a contradiction. Take a pair $(x,y) \in \mathcal{D}$ with $\|x-y\|_1$ minimum, and fix $u_* \in \mathrm{supp}^+(x-y)$ appearing in the definition of $\mathcal{D}$. Then, we have $\|x-y\|_1 \geq 4$, $\mathrm{supp}^+(x-y) \neq \emptyset$, and $\mathrm{supp}^-(x-y) \neq \emptyset$ by Claims 1 and 3.

Take any $u_0 \in \mathrm{supp}^+(x-y-\chi_{u_*})$, and put

$$X = \{v \in \mathrm{supp}^-(x-y) \cup \{0\} \mid x - \chi_{u_*} + \chi_v \in Q\},$$
$$Y = \{v \in \mathrm{supp}^-(x-y) \cup \{0\} \mid y + \chi_{u_0} - \chi_v \in Q\},$$

where $X \neq \emptyset$ by (G-EXC$_0$) (i) and $Y \neq \emptyset$ by (G-EXC$_0$) (ii). Since $(x,y) \in \mathcal{D}$, we have

$$y + \chi_{u_*} - \chi_v \notin Q \qquad (\forall v \in X). \tag{9}$$

It also holds that

$$\text{if } X \cap Y = \emptyset, \text{ then } y + \chi_{u_0} - \chi_v \notin Q \qquad (\forall v \in X). \tag{10}$$

Take any $v_0 \in Y$, where we assume $v_0 \in X \cap Y$ if $X \cap Y \neq \emptyset$. Put $y' = y + \chi_{u_0} - \chi_{v_0} \in Q$.

Claim 4: For $v \in \mathrm{supp}^-(x-y') \cup \{0\}$, if $x - \chi_{u_*} + \chi_v \in Q$ (i.e., $v \in X$), then $y' + \chi_{u_*} - \chi_v \notin Q$ holds.

[Proof of Claim 4] Let $v \in \mathrm{supp}^-(x-y') \cup \{0\}$ be such that $x - \chi_{u_*} + \chi_v \in Q$. Put

$$y'' = y' + \chi_{u_*} - \chi_v = y + \chi_{u_0} + \chi_{u_*} - \chi_{v_0} - \chi_v.$$

We assume, to the contrary, that $y'' \in Q$ and derive a contradiction.

(Case of $X \cap Y \neq \emptyset$) Since $v, v_0 \in X$, we have

$$y + \chi_{u_*} - \chi_v \notin Q, \qquad y + \chi_{u_*} - \chi_{v_0} \notin Q \tag{11}$$

by (9). If $v = 0$ or $v_0 = 0$, then $\|y'' - y\|_1 \leq 3$, and hence (11) contradicts Claim 3. Assume $v \neq 0$ and $v_0 \neq 0$. By (11) and (G-EXC$_0$) (i) applied to y'', y, and u_0, we have $y + \chi_{u_*} - \chi_{v_0} - \chi_v \in Q$. Then, $\|(y + \chi_{u_*} - \chi_{v_0} - \chi_v) - y\|_1 = 3$ and (11) contradicts Claim 2.

(Case of $X \cap Y = \emptyset$) First note that $v \neq v_0$ holds since $v \in X$ and $v_0 \in Y$. Hence, we have $v \neq 0$ or $v_0 \neq 0$ (or both). By (9) and (10), we have

$$y + \chi_{u_*} - \chi_v \notin Q, \qquad y + \chi_{u_0} - \chi_v \notin Q. \tag{12}$$

If $v = 0$ or $v_0 = 0$, then $\|y'' - y\|_1 \leq 3$, and hence (12) contradicts Claim 3. Assume $v \neq 0$ and $v_0 \neq 0$. By (12) and (G-EXC$_0$) (ii) applied to y, y'', and v_0, we have $y + \chi_{u_0} + \chi_{u_*} - \chi_v \in Q$. Then, $\|(y + \chi_{u_0} + \chi_{u_*} - \chi_v) - y\|_1 = 3$ and (12) contradicts Claim 2.

[End of Claim 4]

By Claim 4, we have $(x, y') \in \mathcal{D}$, which is a contradiction to the choice of (x, y) since $u_* \in \mathrm{supp}^+(x - y')$ and $\|x - y'\|_1 \leq \|x - y\|_1 - 1$. This concludes the proof of (G-EXC) for Q. (end)