A Proof of the L-Optimality Criterion Theorem

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An alternative proof is given to the theorem of L-optimality criterion on the basis of a property of an L$^\natural$-convex function labelled (L$^\natural$-APR[$\mathbf{Z}$]).

Theorem 1 (L-optimality criterion [1, Th.5.12], [2, Th.7.14]) *For an* $L^\natural$*-convex function* $g \in \mathcal{L}^\natural[\mathbf{Z} \to \mathbf{R}]$ *and* $p \in \mathrm{dom}\, g$*, we have*

$$g(p) \leq g(q) \quad (\forall\, q \in \mathbf{Z}^V) \iff g(p) \leq g(p \pm \chi_Y) \quad (\forall\, Y \subseteq V). \tag{1}$$

Recall ([1, Th.5.15], [2, Th.7.7]) the following property of an L$^\natural$-convex function g:

> **(L$^\natural$-APR[Z])** For any $p, q \in \mathbf{Z}^V$ with $\mathrm{supp}^+(p-q) \neq \emptyset$, it holds that
> $$g(p) + g(q) \geq g(p - \chi_X) + g(q + \chi_X),$$
> where $X = \arg\max_{v \in V} \{p(v) - q(v)\}$.

This implies the following:

$$\begin{aligned} & p, q \in \mathrm{dom}\, g,\ g(p) > g(q) \implies \\ & g(p) > \min\left(\min_{Y \subseteq \mathrm{supp}^+(q-p)} g(p + \chi_Y),\ \min_{Z \subseteq \mathrm{supp}^-(q-p)} g(p - \chi_Z) \right), \end{aligned} \tag{2}$$

as stated below. Theorem 1 ($\Longleftarrow$) is an immediate consequence of this, whereas ($\Longrightarrow$) is obvious.

Proposition 2 *An* $L^\natural$*-convex function* $g \in \mathcal{L}^\natural[\mathbf{Z} \to \mathbf{R}]$ *satisfies* (2).

(Proof) If $\mathrm{supp}^+(q-p)$ is nonempty, (L$^\natural$-APR[$\mathbf{Z}$]) applied to (q, p) implies the existence of $Y_1 \subseteq \mathrm{supp}^+(q-p)$ such that

$$g(q) \geq [g(p + \chi_{Y_1}) - g(p)] + g(q_2),$$

where $q_2 = q - \chi_{Y_1}$. If $\mathrm{supp}^+(q_2 - p)$ is nonempty, (L$^\natural$-APR[**Z**]) applied to (q_2, p) implies the existence of $Y_2 \subseteq \mathrm{supp}^+(q_2 - p)$ such that

$$g(q_2) \geq [g(p + \chi_{Y_2}) - g(p)] + g(q_3),$$

where $q_3 = q_2 - \chi_{Y_2} = q - \chi_{Y_1} - \chi_{Y_2}$. Repeating this, we obtain $q' = q - \sum_{i=1}^{m} \chi_{Y_i}$ with $Y_i \subseteq \mathrm{supp}^+(q - p)$ $(i = 1, \ldots, m)$ such that $q'(v) = p(v)$ for $v \in \mathrm{supp}^+(q - p)$, $q'(v) = q(v)$ for $v \in V \setminus \mathrm{supp}^+(q - p)$, and

$$g(q) \geq g(q') + \sum_{i=1}^{m} [g(p + \chi_{Y_i}) - g(p)]. \tag{3}$$

By the similar procedure starting with (p, q') we obtain $Z_j \subseteq \mathrm{supp}^-(q - p)$ $(j = 1, \ldots, l)$ such that $p = q' + \sum_{j=1}^{l} \chi_{Z_j}$ and

$$g(q') \geq g(p) + \sum_{j=1}^{l} [g(p - \chi_{Z_j}) - g(p)]. \tag{4}$$

Adding (3) and (4) we obtain

$$g(q) \geq g(p) + \sum_{i=1}^{m} [g(p + \chi_{Y_i}) - g(p)] + \sum_{j=1}^{l} [g(p - \chi_{Z_j}) - g(p)].$$

Since $g(q) < g(p)$, we have that $g(p + \chi_{Y_i}) - g(p) < 0$ for some i or $g(p - \chi_{Z_j}) - g(p) < 0$ for some j. (END)

NOTE: Compare Proposition 2 above with the similar statement for an M-convex function f given in [1, Prop.4.15], [2, Prop.6.23]:

$$\begin{aligned} & x, y \in \mathrm{dom}\, f,\ f(x) > f(y) \\ & \Longrightarrow\ f(x) > \min_{u \in \mathrm{supp}^+(x-y)} \ \min_{v \in \mathrm{supp}^-(x-y)} f(x - \chi_u + \chi_v). \end{aligned} \tag{5}$$

References

[1] K. Murota: *Discrete Convex Analysis—An Introduction* (in Japanese), Kyoritsu Publishing Co., Tokyo, 2001.

[2] K. Murota: *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications, Vol. 10, Society for Industrial and Applied Mathematics, Philadelphia, 2003.