RIMS Summer School (COSS 2018), Kyoto, July 2018

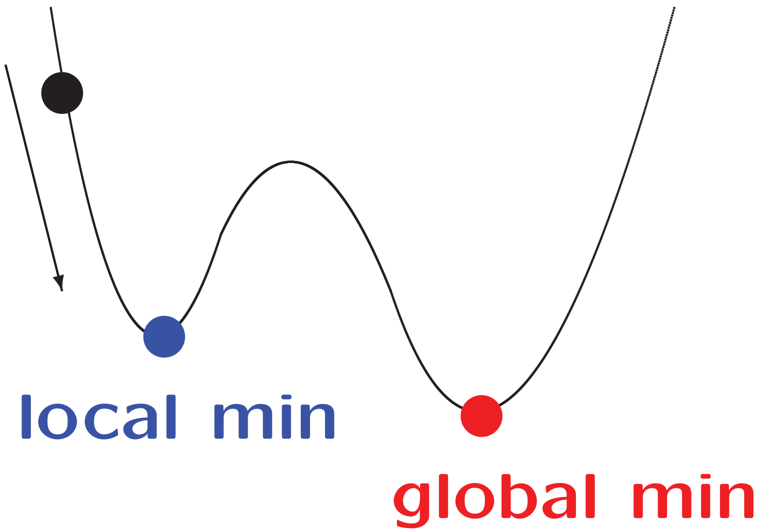
Discrete Convex Analysis I:

Concepts of Discrete Convex Functions

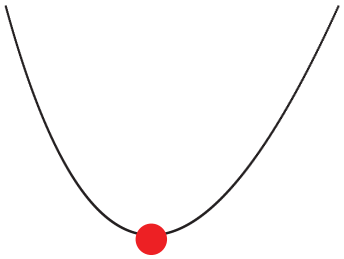
Kazuo Murota

(Tokyo Metropolitan University)

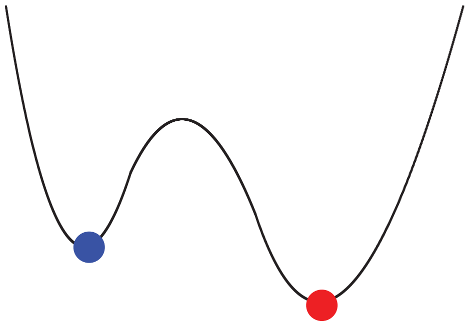
Minimization and Convexity



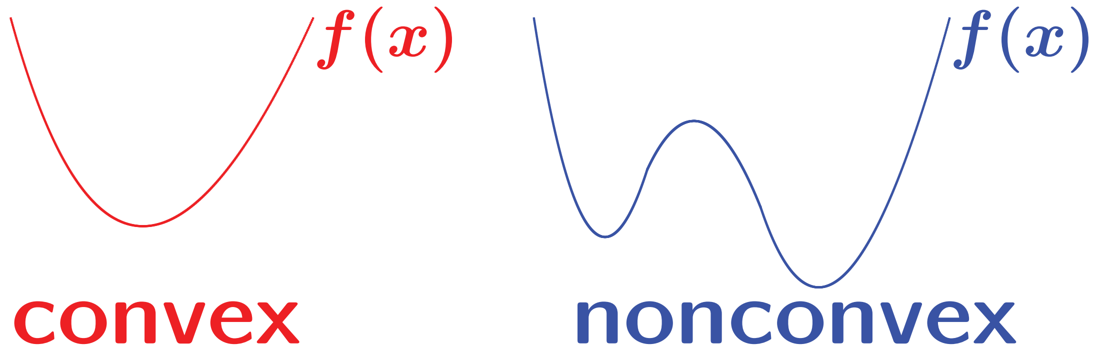
convex



nonconvex

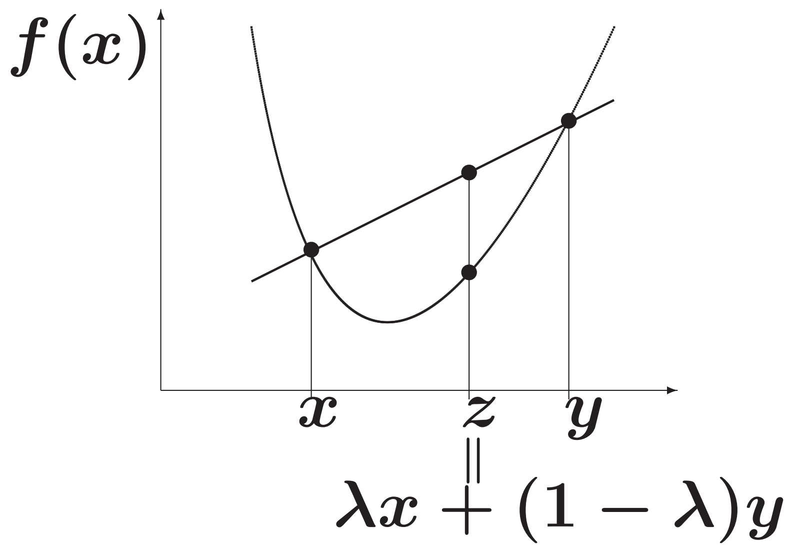


Convex Function



$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex $\iff$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \qquad (0 < \forall\ \lambda < 1)$$



$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

Contents of Part I

Concepts of Discrete Convex Functions

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

C1.
Univariate
Discrete Convex Functions

(Ingredients of convex analysis)

Definition of "Convex" Function

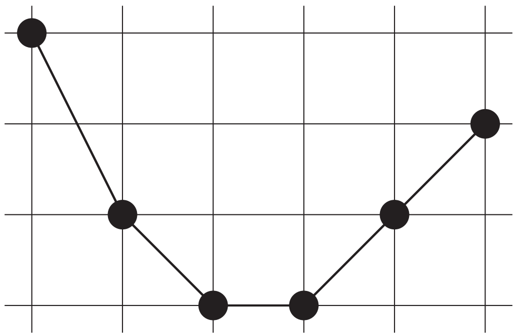
$f : \mathbb{Z} \to \overline{\mathbb{R}}$ $\qquad\qquad$ $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

$f(x-1) + f(x+1) \geq 2f(x)$

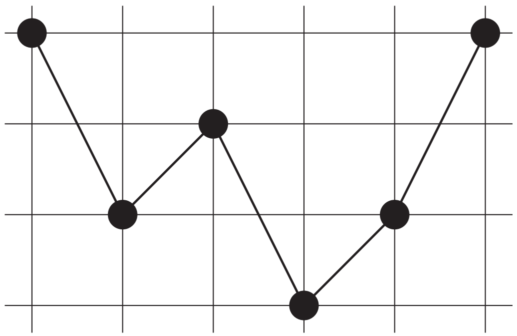
$\iff \quad f(x) + f(y) \geq f(x+1) + f(y-1) \qquad (x < y)$

$\iff$ f is **convex-extensible**, i.e.,

$\exists$ convex $\overline{f} : \mathbb{R} \to \overline{\mathbb{R}}$ s.t. $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z})$



convex



non-convex

Local vs Global Optimality

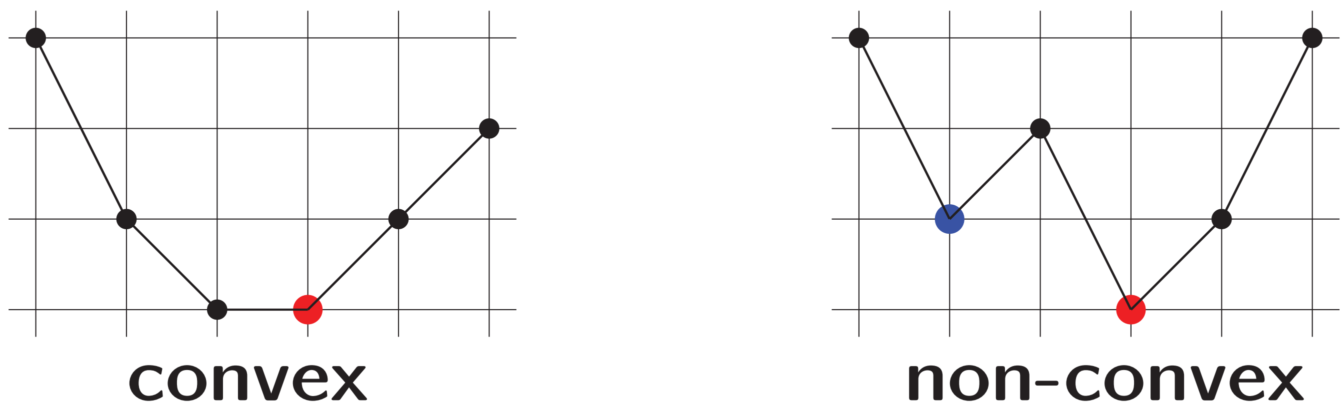
$f : \mathbb{Z} \to \overline{\mathbb{R}}$

Theorem:

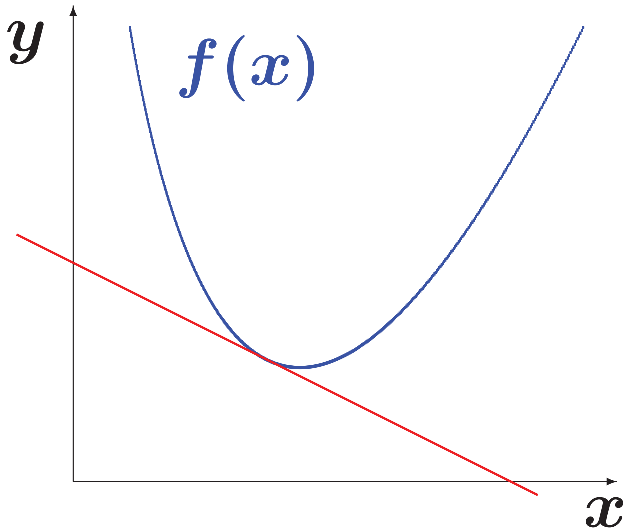
x^*: global opt (min)

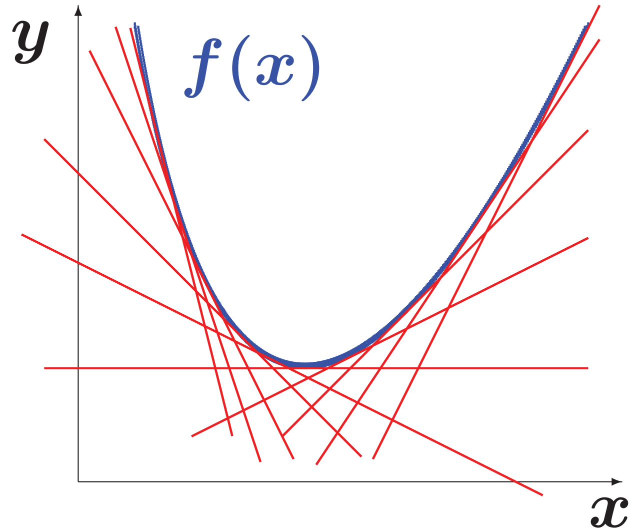
$\iff$ x^*: local opt (min)

$$f(x^*) \leq \min\{f(x^* - 1),\ f(x^* + 1)\}$$

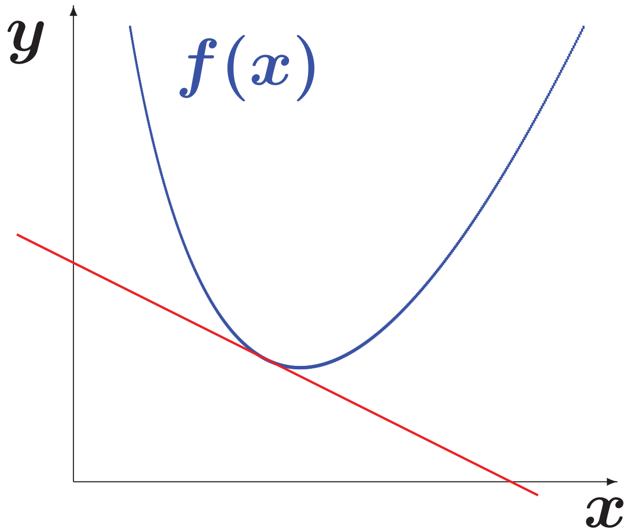


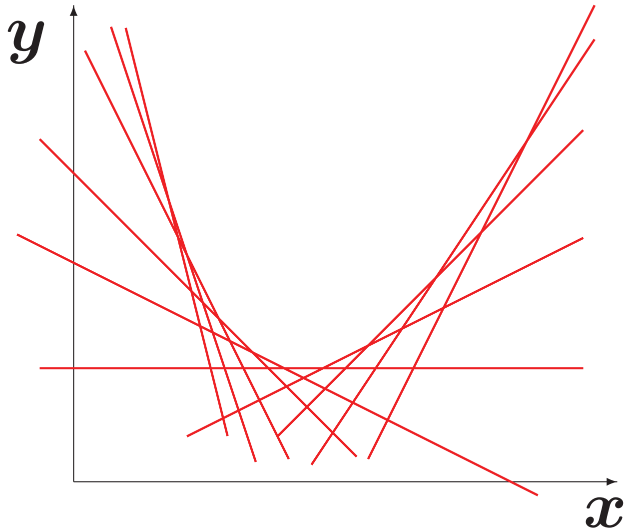
Legendre Transformation (Intuition)



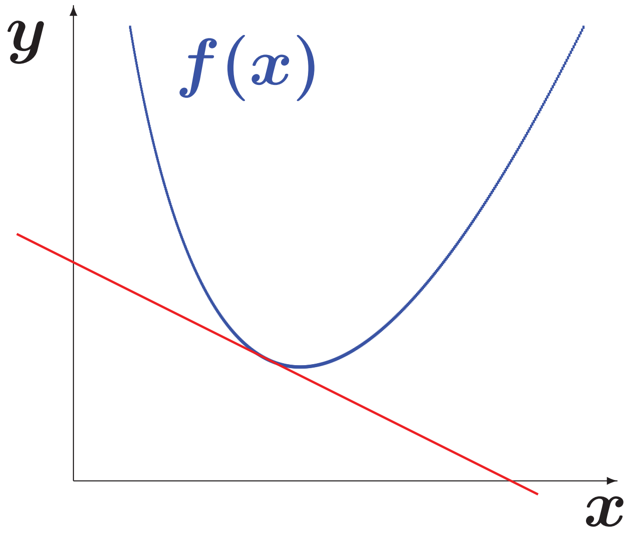


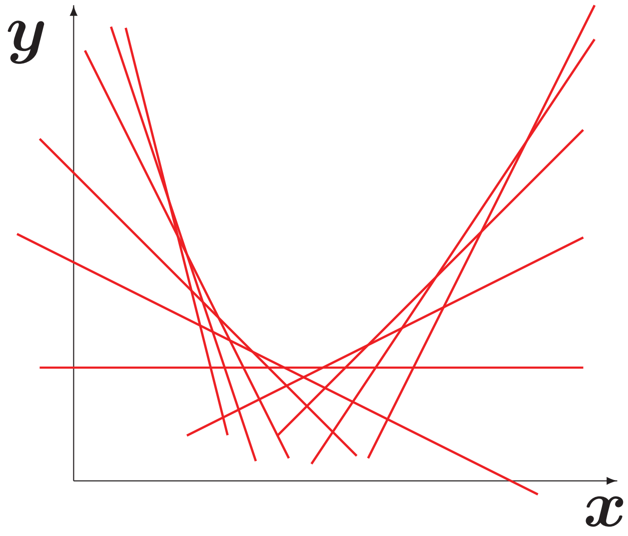
Legendre Transformation (Intuition)

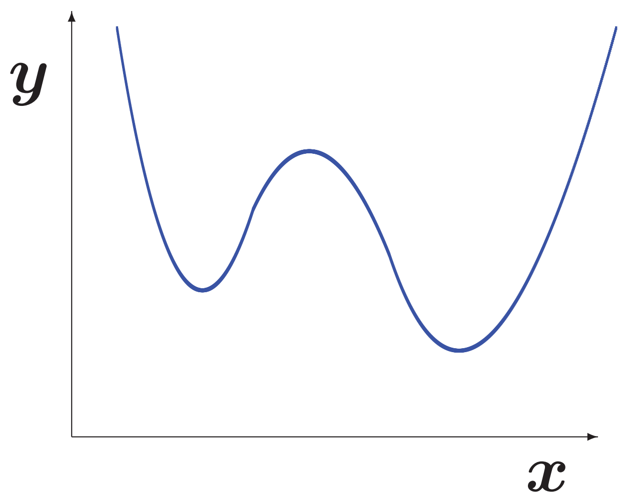


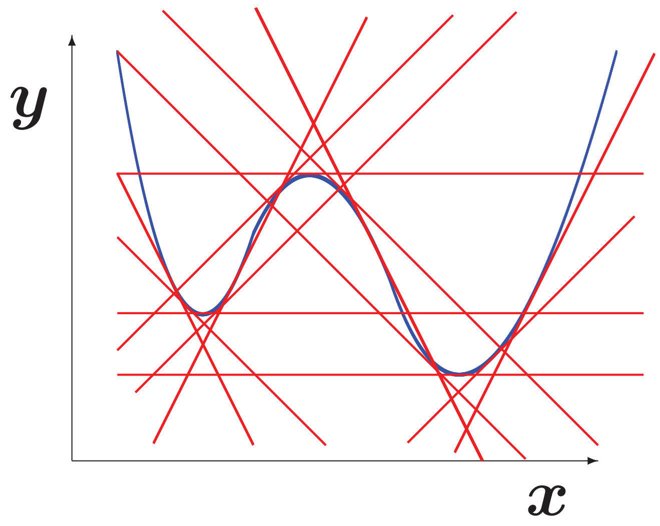


Legendre Transformation (Intuition)







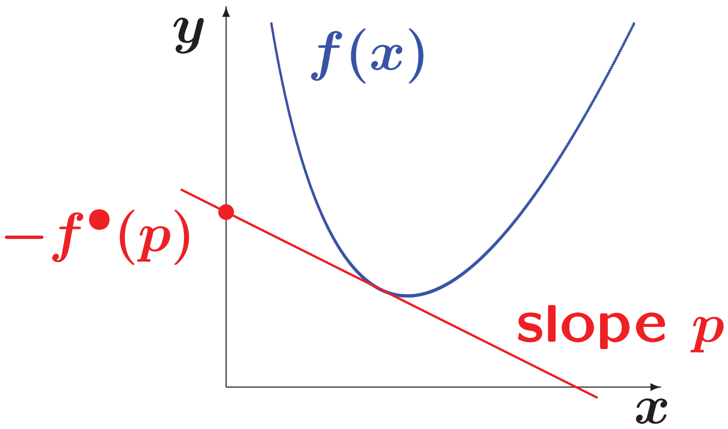


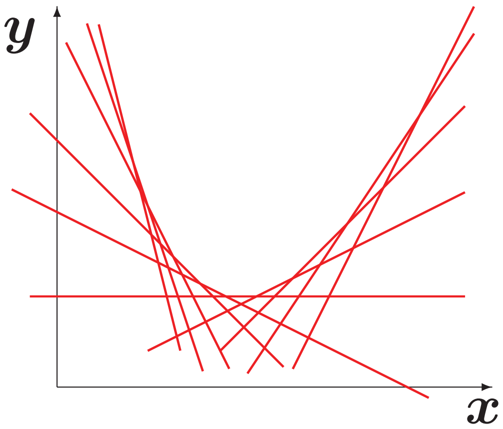
Legendre Transformation

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$ (integer-valued)

Define **discrete Legendre transform** of f by

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z})$$





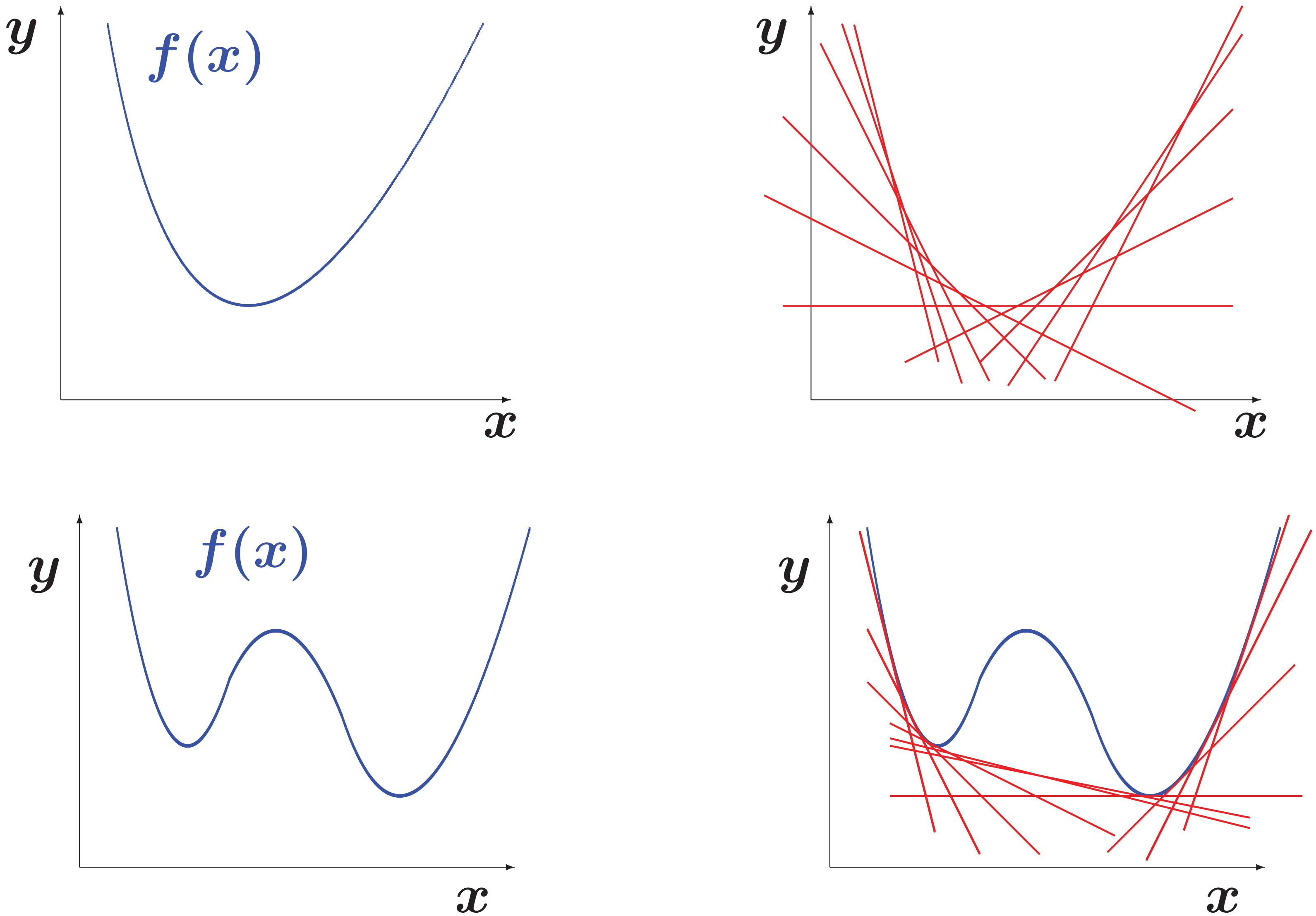
Theorem:

(1) $f^{\bullet}$ is $\mathbb{Z}$-valued convex function, $f^{\bullet} : \mathbb{Z} \to \overline{\mathbb{Z}}$

(2) $(f^{\bullet})^{\bullet} = f$ (biconjugacy)

Legendre Transformation

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\}$$



Conjugacy for Quadratic Function

$f(x) = x^2$

$\mathbb{R}$: $f^{\bullet}(p) = \max\{px - x^2 \mid x \in \mathbb{R}\} = \frac{1}{4}p^2$

$f^{\bullet\bullet}(x) = \max\{px - \frac{1}{4}p^2 \mid p \in \mathbb{R}\} = x^2$

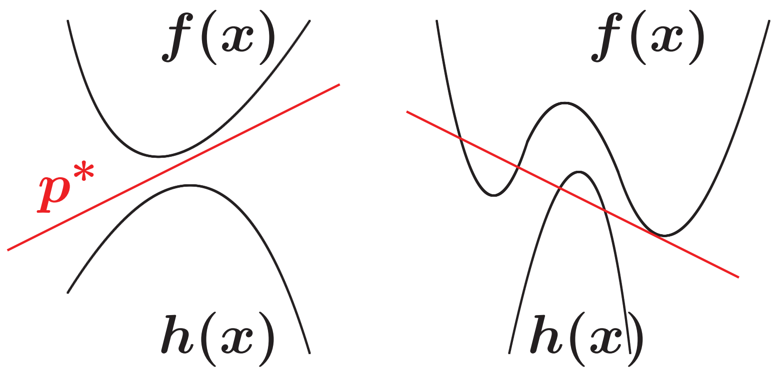
$\mathbb{Z}$: $f^{\bullet}(p) = \max\{px - x^2 \mid x \in \mathbb{Z}\} = \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil$

$$f^{\bullet\bullet}(x) = \max\left\{px - \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil \mid p \in \mathbb{Z}\right\} = x^2$$

Separation Theorem

$f : \mathbb{Z} \to \overline{\mathbb{R}}$
convex

$h : \mathbb{Z} \to \underline{\mathbb{R}}$
concave



Theorem (Discrete Separation Theorem)

(1) $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}) \Rightarrow \exists\, \alpha^* \in \mathbb{R}, \ \exists\, p^* \in \mathbb{R}$:

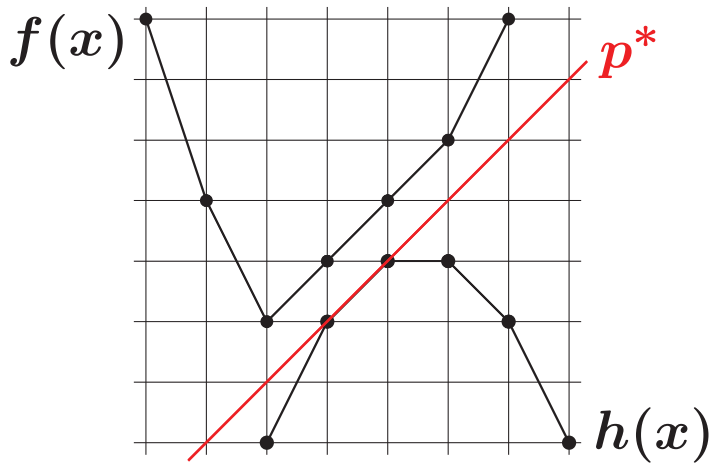
$$f(x) \geq \alpha^* + p^* x \geq h(x) \quad (\forall x \in \mathbb{Z})$$

(2) f, h: integer-valued $\Rightarrow \alpha^* \in \mathbb{Z}, \ p^* \in \mathbb{Z}$

Separation Theorem

$f : \mathbb{Z} \to \overline{\mathbb{R}}$
convex

$h : \mathbb{Z} \to \underline{\mathbb{R}}$
concave



Theorem (Discrete Separation Theorem)

(1) $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}) \Rightarrow \ \exists\, \alpha^* \in \mathbb{R}, \ \ \exists\, p^* \in \mathbb{R}$:

$$f(x) \geq \alpha^* + p^* x \geq h(x) \quad (\forall x \in \mathbb{Z})$$

(2) f, h: integer-valued $\Rightarrow \alpha^* \in \mathbb{Z}, \ \ p^* \in \mathbb{Z}$

Fenchel Duality (Min-Max)

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$: convex, $\quad h : \mathbb{Z} \to \underline{\mathbb{Z}}$: concave

Legendre transforms:

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\}$$

$$h^{\circ}(p) = \inf\{px - h(x) \mid x \in \mathbb{Z}\}$$

Theorem:

$$\inf_{x \in \mathbb{Z}} \{f(x) - h(x)\} = \sup_{p \in \mathbb{Z}} \{h^{\circ}(p) - f^{\bullet}(p)\}$$

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **univariate**
discrete convex functions

C2.

Classes of Discrete Convex Functions

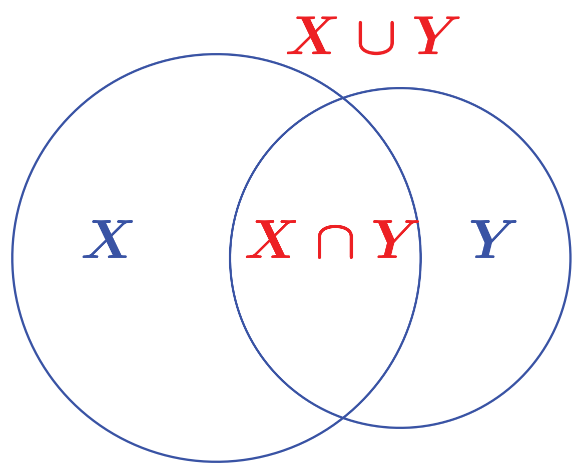
Submodular Function

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Set function $\rho : 2^V \to \overline{\mathbb{R}}$ **is submodular**

$\Longleftrightarrow$

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



cf. $|X| + |Y| = |X \cup Y| + |X \cap Y|$

Set function $\Longleftrightarrow$ **Function on** $\{0,1\}^n$

Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

(Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

min $\Rightarrow$ polynomial, max $\Rightarrow$ exponential

- Convex extension (Lovász)

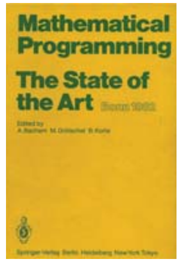
set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

discrete separation, Fenchel min-max

Submodular set functions
= Convexity + Discreteness

Submodular functions and convexity



11th Math.Prog.Symp, Bonn, 1982

L. Lovász
Eötvös Loránd University, Department of Analysis I, Múzeum krt. 6–8, H-1088 Budapest, Hungary

---

– Convex functions occur in many mathematical models in economy, engineering, and other sciencies. Convexity is a very natural property of various functions and domains occuring in such models; quite often the only non-trivial property which can be stated in general.

– Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.

– Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.

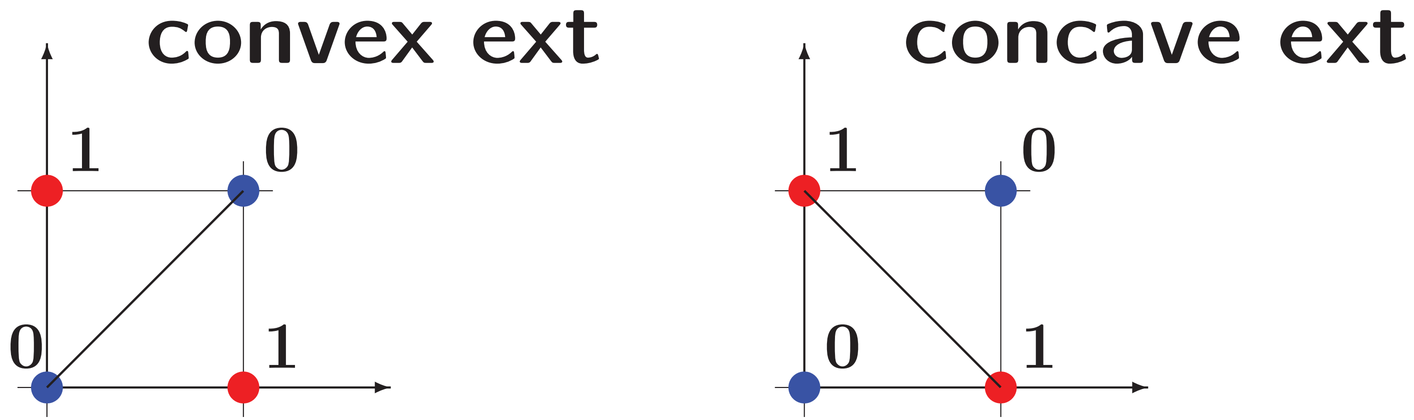
– There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.

Set Function and Extensions

Set function $\iff$ Function on $\{0,1\}^n$

$$\rho(X) \quad = \quad \hat{\rho}(\chi_X)$$

Every set function $\rho : \{0,1\}^n \to \mathbb{R}$ can be extended to convex/concave function



cf. Lovász extension

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **submodular set functions**

Classes of Discrete Convex Functions

1. Submodular set fn (on $\{0,1\}^n$)
1. Separable-convex fn on $\mathbb{Z}^n$
1. Integrally-convex fn on $\mathbb{Z}^n$

2. L-convex (L$^\natural$-convex) fn on $\mathbb{Z}^n$
2. M-convex (M$^\natural$-convex) fn on $\mathbb{Z}^n$

3. M-convex fn on jump systems
3. L-convex fn on graphs

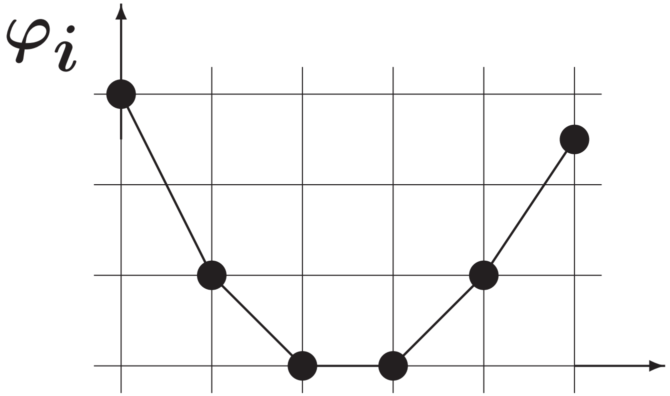
Separable-convex Function

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is **separable-convex**

$\iff$

$$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$$

φ_i: univariate convex



Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **separable** **discrete convex functions**

Discrete Convex Functions

1. submodular (set fn)	✓
1. separable -conv	✓
1. integrally -conv	
2. L-conv($\mathbb{Z}^n$)	
2. M-conv($\mathbb{Z}^n$)	
3. M-conv(jump)	
3. L-conv(graph)	

Some History

1935 Matroid — Whitney, Nakasawa

1965 Submodular function — Edmonds

1969 Convex network flow (electr.circuit) — Iri

1982 Submodularity and convexity — Frank, Fujishige, Lovász

1990 Valuated matroid — Dress–Wenzel
Integrally convex fn — Favati–Tardella

1996 Discrete convex analysis — Murota
$M/L/M^\natural/L^\natural$ — M.-Shioura, Fujishige-M.

2000 Submodular minimization algorithm — Iwata–Fleischer–Fujishige, Schrijver

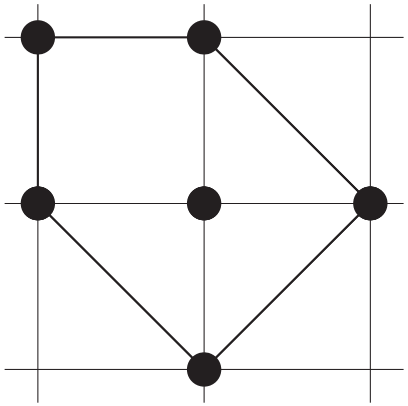
2006 M-convex fn on jump system — Murota

2012 L-convex fn on graph — Hirai, Kolmogorov

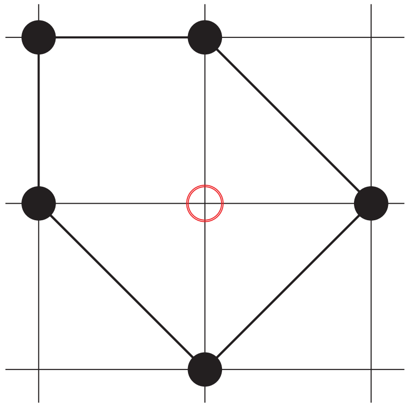
Motivations/Applications/Connections

1. submodular	MANY problems graph cut, convex game
1. separable-conv	MANY problems min-cost flow, resource allocation
1. integrally-conv	economics, game
2. L-conv ($\mathbb{Z}^n$)	network tension, image processing OR (inventory, scheduling)
2. M-conv ($\mathbb{Z}^n$)	network flow, matching economics (game, auction) mixed polynomial matrix
3. M-conv (jump)	deg sequence, (2-)matching polynomial (half-plane property)
3. L-conv (graph)	multiflow, multifacility location

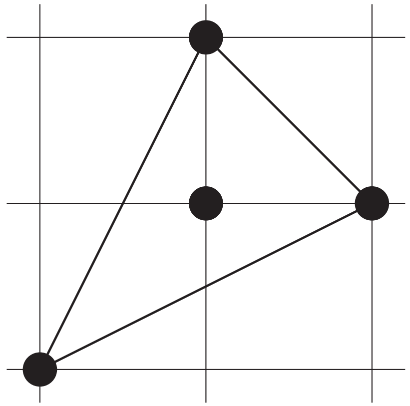
Integrally Convex Set $\subseteq \mathbb{Z}^n$



YES

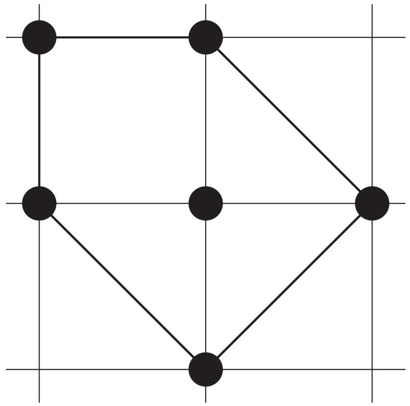


NO

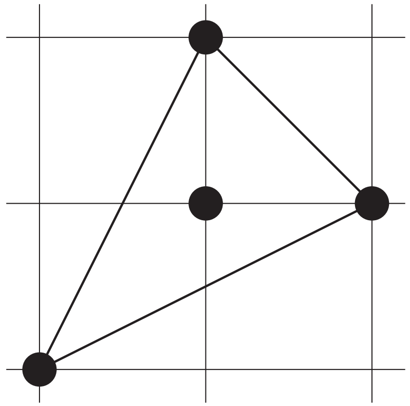


NO

Integrally Convex Set

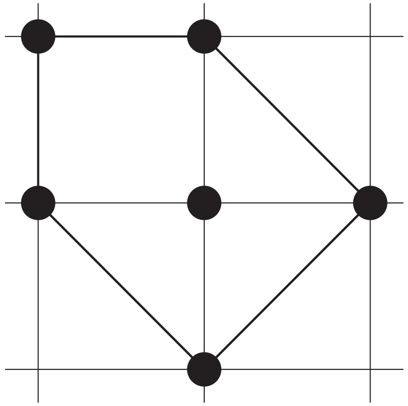


YES

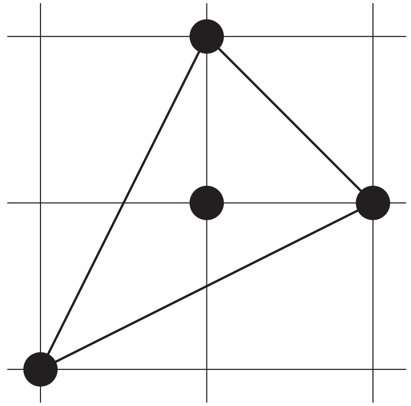


NO

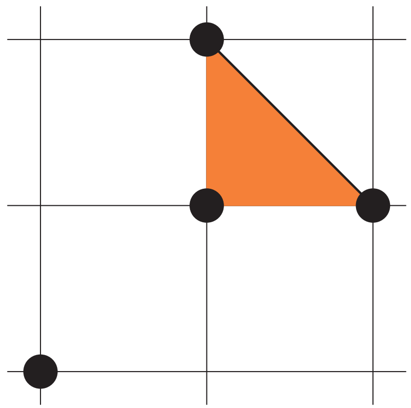
Integrally Convex Set



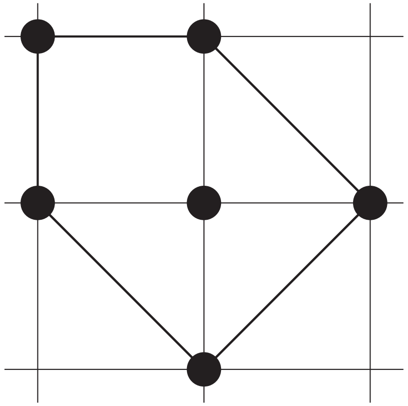
YES



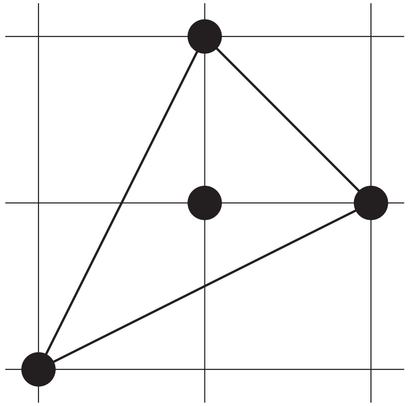
NO



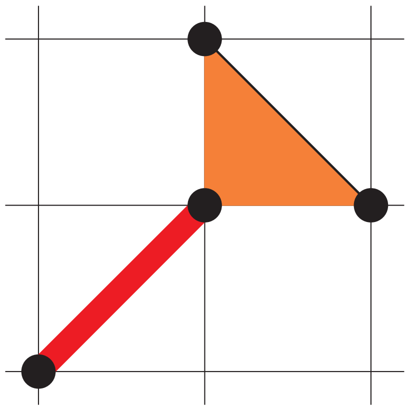
Integrally Convex Set



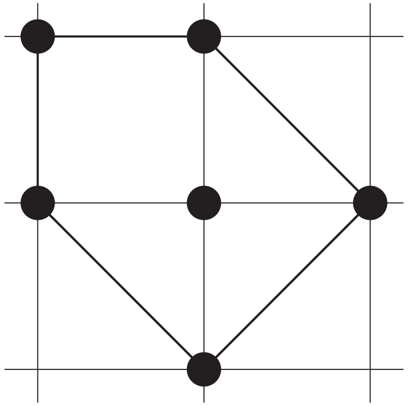
YES

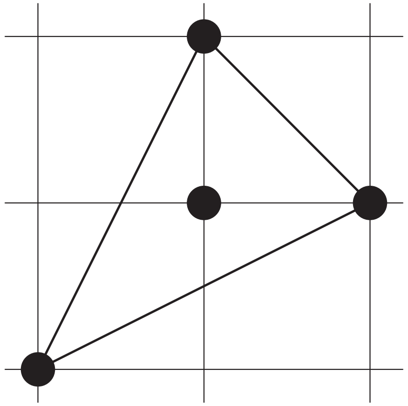


NO



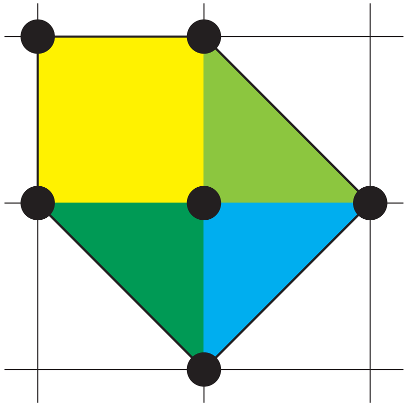
Integrally Convex Set

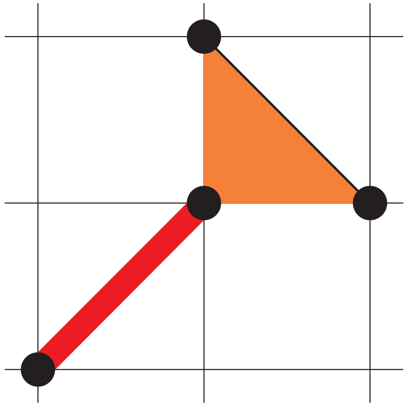




YES

NO

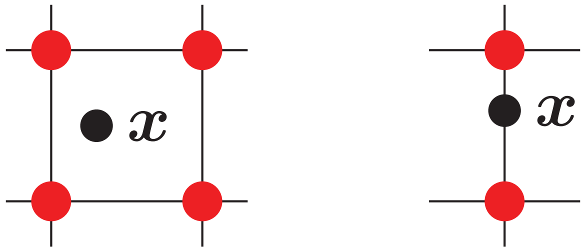




Integrally Convex Function

(Favati-Tardella 1990)

$$N(x) = \{ y \in \mathbb{Z}^n \mid \| x - y \|_\infty < 1 \} \qquad (x \in \mathbb{R}^n)$$

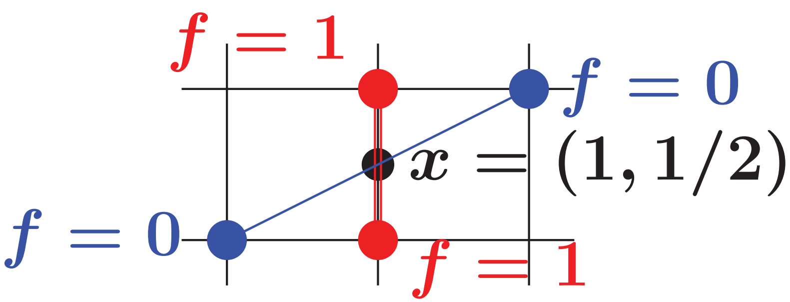


Local convex extension:

$$\tilde{f}(x) = \sup_{p,\alpha} \{ \langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq f(y) \ (\forall y \in N(x)) \}$$

Def: f is integrally convex $\iff$ $\tilde{f}$ is convex

Ex: $f(x_1, x_2) = |x_1 - 2x_2|$ is NOT integrally convex



$\tilde{f}(x) = 1$

$\overline{f}(x) = 0$

Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. biconjugacy (Legendre transform $\times 2$)

hold, but

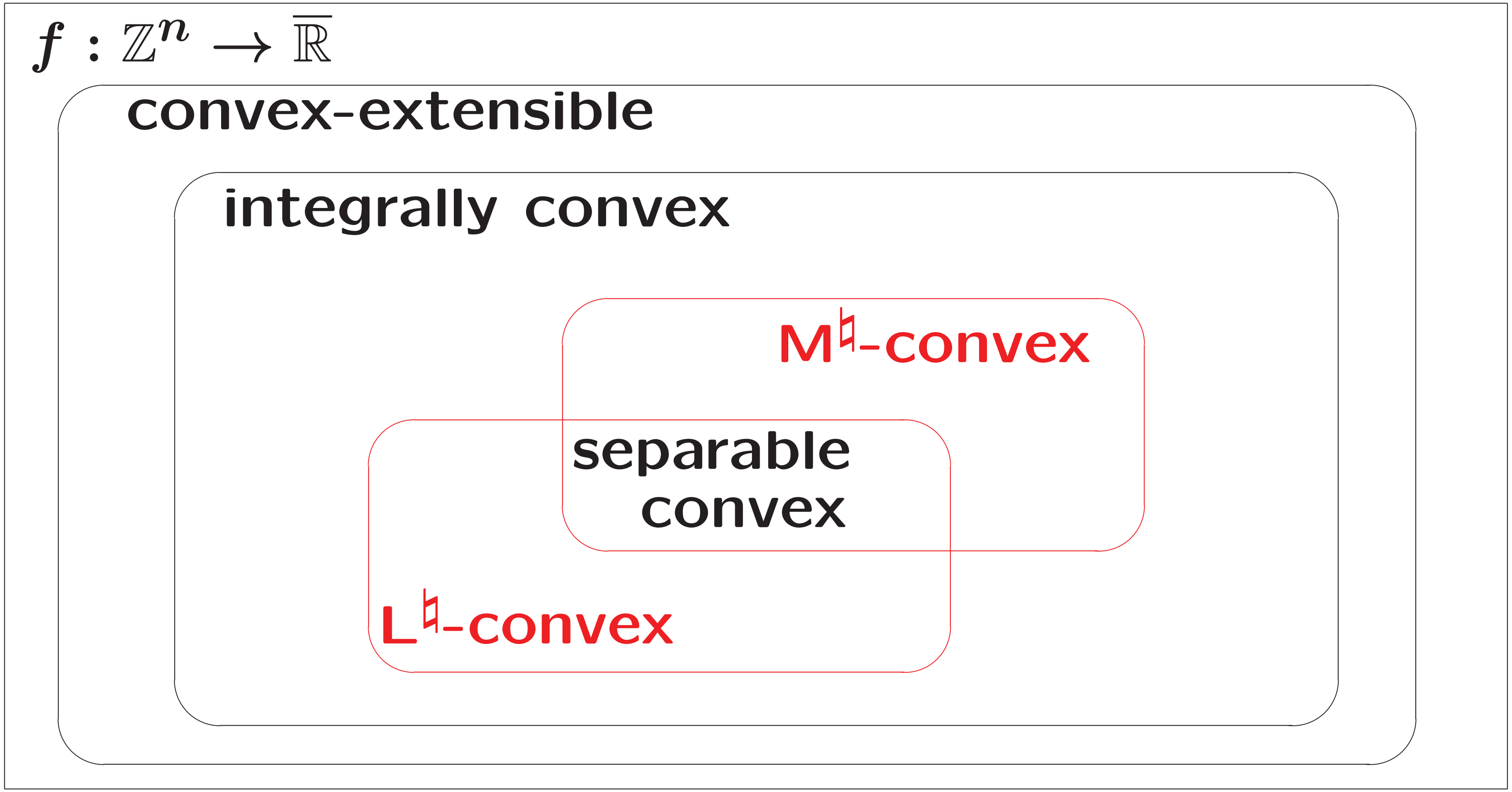
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

fail for integrally convex functions

Discrete Convex Functions

1. submodular (set fn)	✓
1. separable -conv	✓
1. integrally -conv	✓
2. L-conv($\mathbb{Z}^n$)	
2. M-conv($\mathbb{Z}^n$)	
3. M-conv(jump)	
3. L-conv(graph)	

Classes of Discrete Convex Functions



$\mathrm{M}^\natural \cap \mathrm{L}^\natural =$ separable

C3.

L-convex Functions

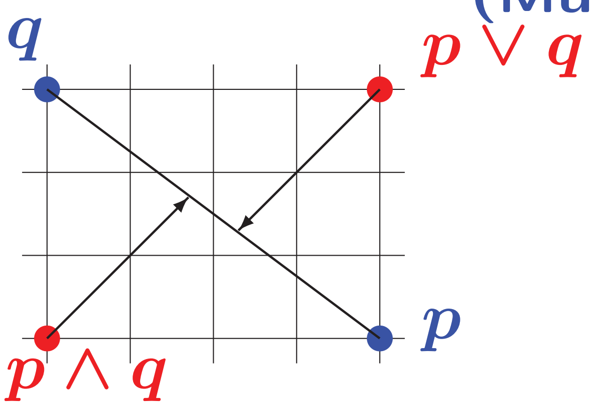
L-convex Function (L = Lattice)

(Murota 98)

$g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

$p \vee q$ compnt-max

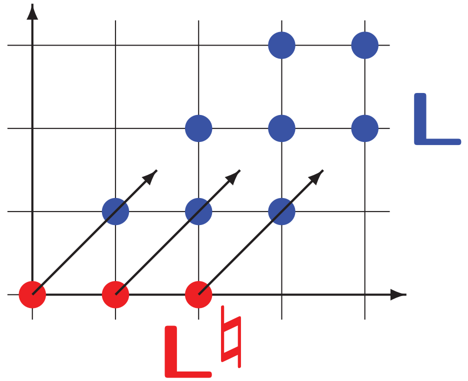
$p \wedge q$ compnt-min



Def: g is L-convex $\iff$

- Submodular: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$
- Translation: $\exists r,\ \forall p$: $g(p + \mathbf{1}) = g(p) + r$

$$\mathbf{1} = (1, 1, \ldots, 1)$$



L♮-convexity from Submodularity

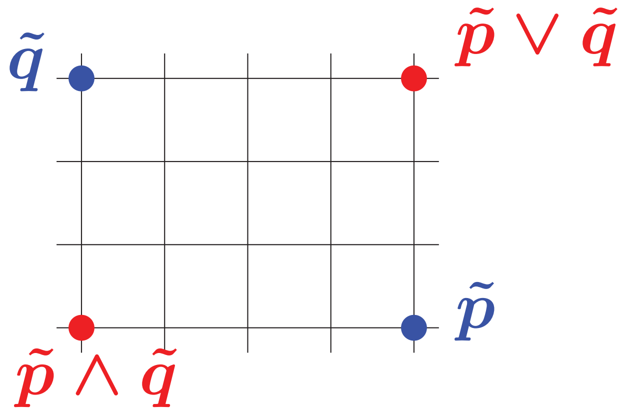
(Murota 98, Fujishige–Murota 2000)

$g : \mathbb{Z}^n \to \mathbb{R}$ **L♮-convex** $\iff$

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in (p_0, p)

$\tilde{g} : \mathbb{Z}^{n+1} \to \mathbb{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1)$

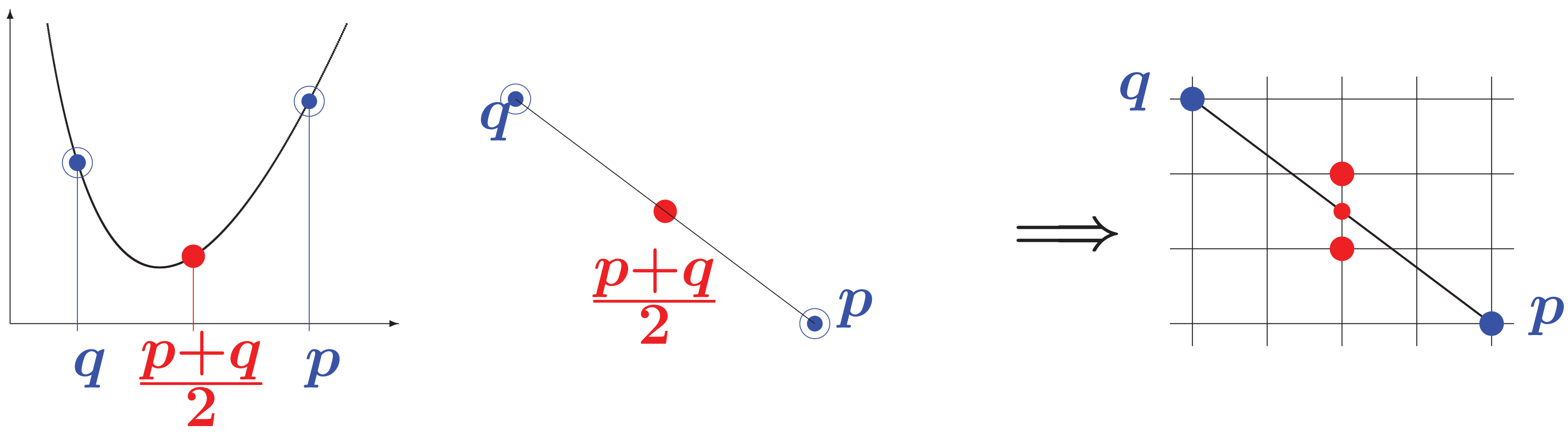
$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}\,(\tilde{p} \vee \tilde{q}) + \tilde{g}\,(\tilde{p} \wedge \tilde{q})$



$\mathrm{L}_{n+1} \simeq \mathrm{L}^{\natural}_{n} \supsetneqq \mathrm{L}_n$

L♮-convexity from Mid-pt-convexity

(Favati-Tardella 1990, Fujishige–Murota 2000)



Mid-point convex ($g : \mathbb{R}^n \to \mathbb{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

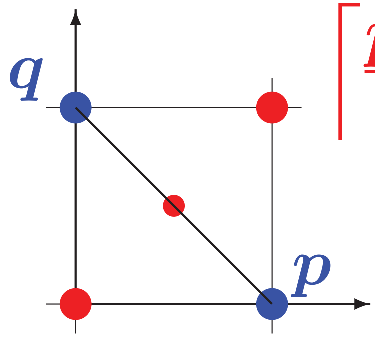
$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathbb{Z}^n \to \mathbb{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

L♮-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$



$$\left\lceil \frac{p+q}{2} \right\rceil = p \vee q$$

$$\left\lfloor \frac{p+q}{2} \right\rfloor = p \wedge q$$

Discrete mid-pt convexity:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

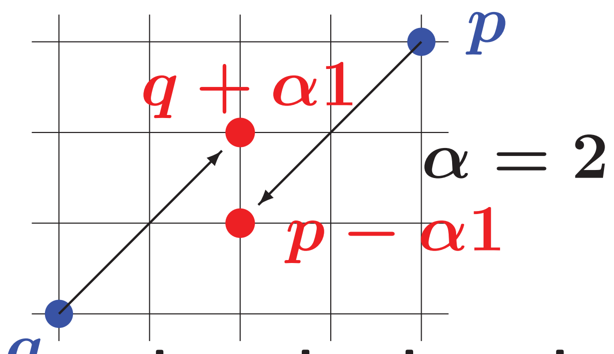
$\iff$ Submodularity:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

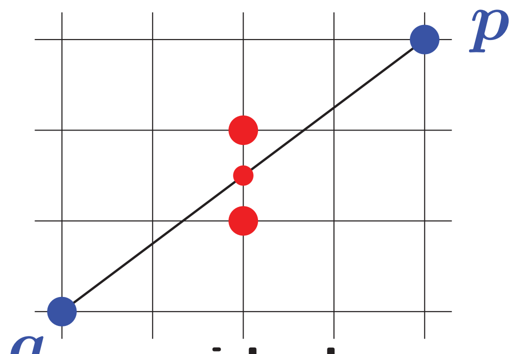
Translation Submodularity (L$\natural$)

$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1))$$

$(\alpha \geq 0)$



transl submod



mid-pt convex

$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

(Fujishige-Murota 00)

$\Leftrightarrow$ translation submodular

(Fujishige-Murota 00)

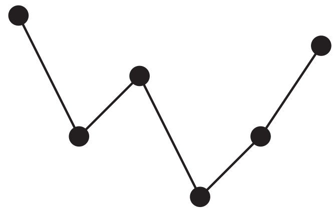
$\Leftrightarrow$ discrete mid-pt convex

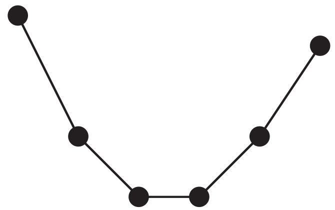
(Favati-Tardella 90)

$\Leftrightarrow$ submod. integ. convex

Rem: L$\natural$-convex vs Submodular

$n = 1$





Fact 1: **Every** $g : \mathbb{Z} \to \mathbb{R}$ **is submodular**

Fact 2: $g : \mathbb{Z} \to \mathbb{R}$ **is L$\natural$-convex**

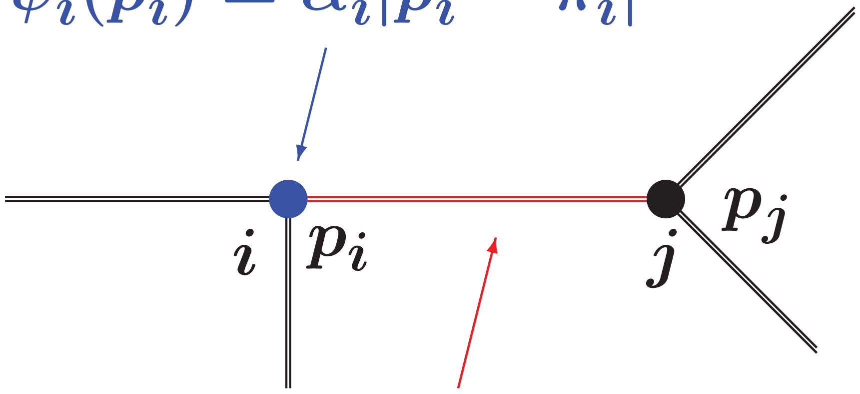
$\iff g(p-1) + g(p+1) \geq 2g(p)$ **for all** $p \in \mathbb{Z}$

Typical L$^\natural$-convex Function: Energy Function

π_i: given

$\alpha_i, \alpha_{ij} \geq 0$

$\psi_i(p_i) = \alpha_i |p_i - \pi_i|^2$



$\psi_{ij}(p_i - p_j) = \alpha_{ij} |p_i - p_j|^2$

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$ is L$^\natural$-convex

ψ_i, ψ_{ij}: any univariate convex functions

L♮-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L♮-convex

$$\Leftrightarrow a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Energy function: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathbb{R}}$

$\Leftrightarrow \rho(X) = g(\chi_X)$ for some L♮-convex g

Multimodular: $h : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L♮-convex g

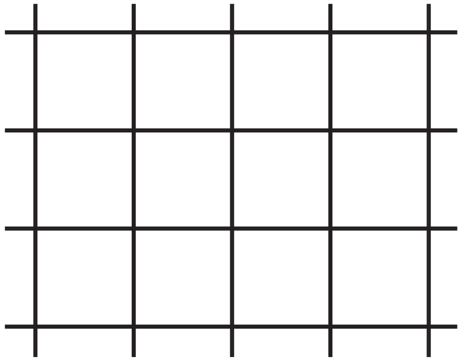
Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

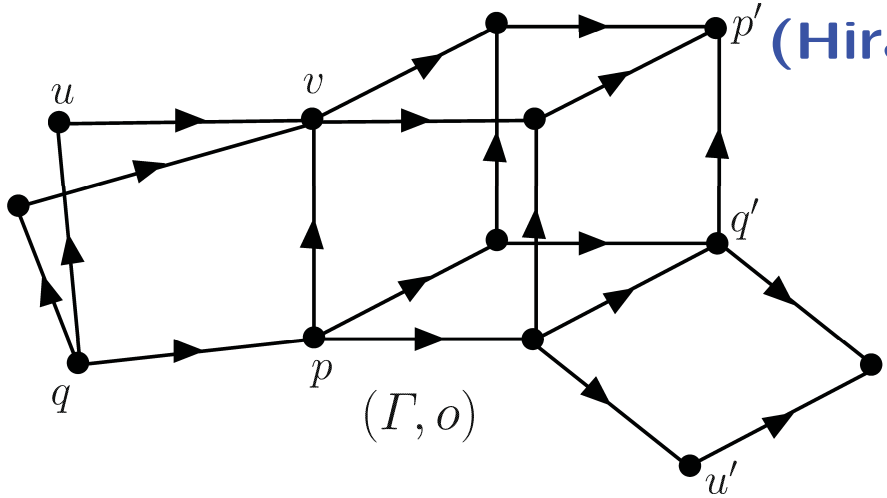
hold for **L-convex functions**

$\Rightarrow$ Part II

L-convex Function on Graphs



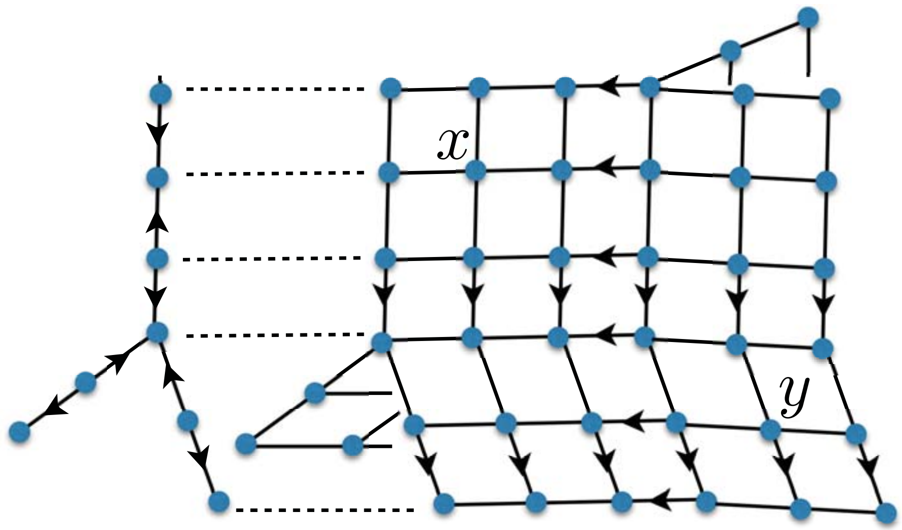
integer lattice



(Hirai 12)

oriented modular graph

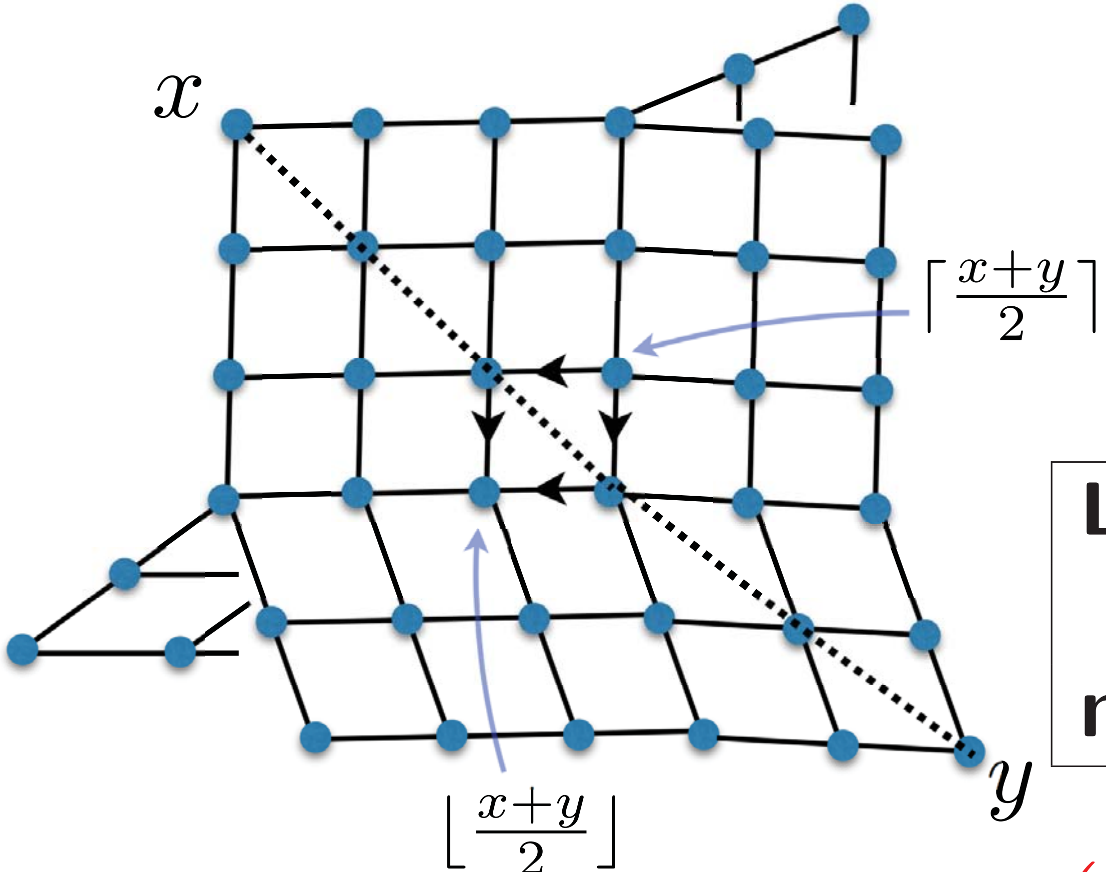
direct product of trees:



(Kolmogorov 11)
(Huber-Kolmogorov 12)

Mid-point Convexity on Tree Products

(Hirai 13,15)



L-convex
|| (def)
mid-point convex

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$$

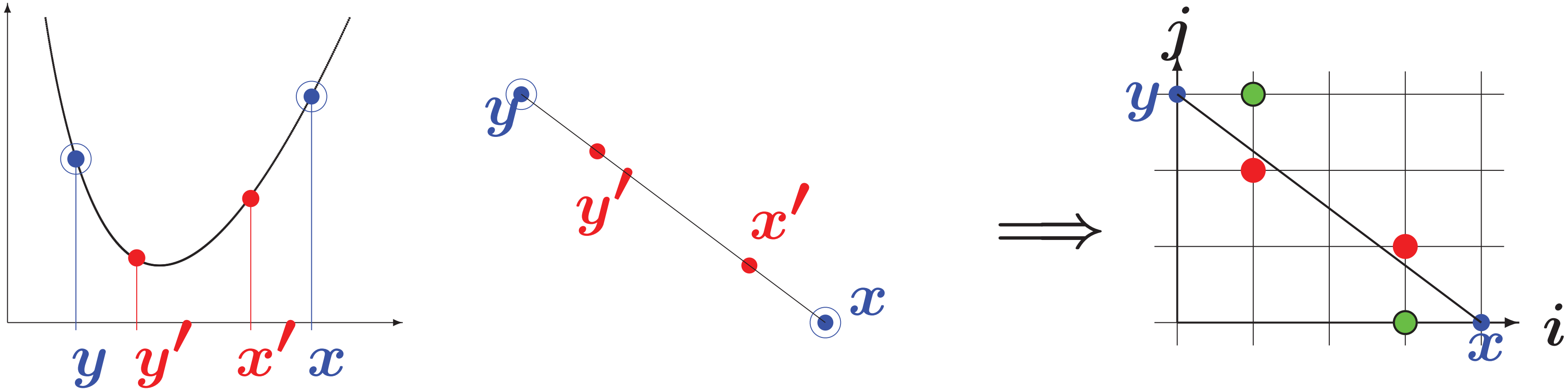
- submodular on (rooted) trees (Kolmogorov 11)
- k-submodular (Huber-Kolmogorov 12)

C4.

M-convex Functions

M$^\natural$-convexity from Equi-dist-convexity

(Murota 1996, Murota–Shioura 1999)



Equi-distance convex ($f : \mathbb{R}^n \to \mathbb{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange ($f : \mathbb{Z}^n \to \mathbb{R}$)** $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \Big[f(x - e_i) + f(y + e_i), \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

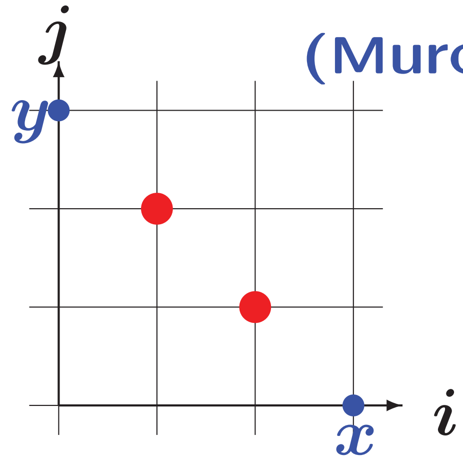
M$^\natural$-convex function $\qquad$ (M = Matroid)

M-convex Function (M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector

(Murota 96)

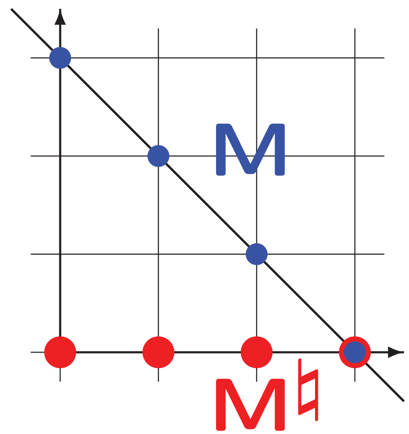


Def: f is M-convex

$\iff \quad \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom}\, f \subseteq$ const-sum hyperplane



$$\mathrm{M}_{n+1} \simeq \mathrm{M}^{\natural}_n \supsetneqq \mathrm{M}_n$$

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow \ a_{ij} \geq 0, \qquad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i,j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, \ldots.\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

M♮-concave Functions from Matroids

Matroid rank: $f(X) = r(X)$ (rank of X) (Fujishige 05)

Matroid rank sum: $f(X) = \sum \alpha_i r_i(X)$

$r_i \leftarrow r_{i+1}$ (strong quotient), $\alpha_i \geq 0$ (Shioura 12)

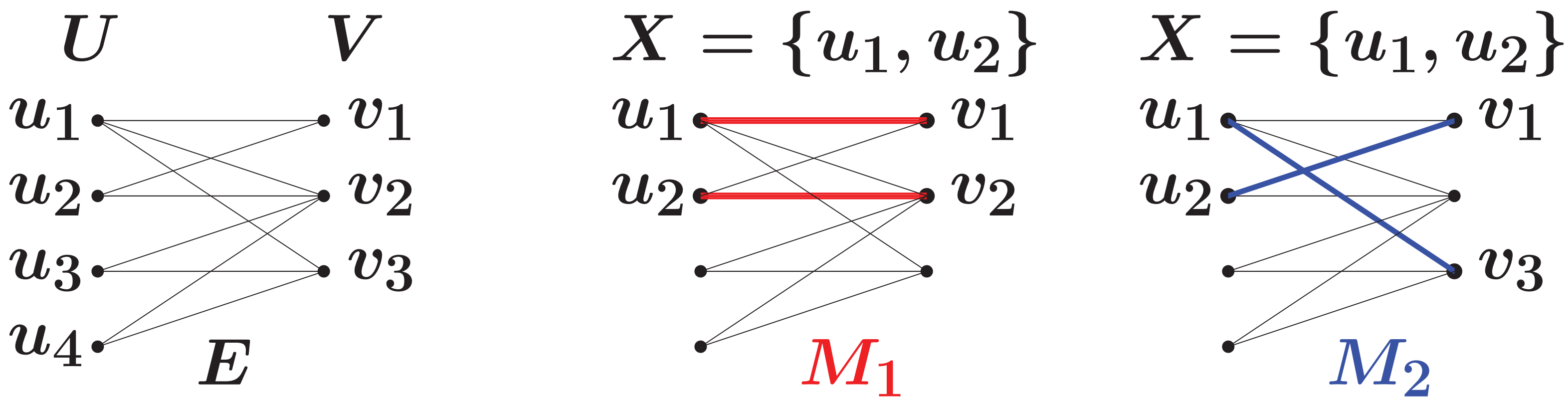
Weighted matroid: w: weight vector

$f(X) = \max\{w(Y) \mid Y\text{: indep} \subseteq X\}$ (Shioura 12)

Valuated matroid: $\omega : 2^V \to \mathbb{R}$

$\Leftrightarrow \omega(X) = f(\chi_X)$ for some M-concave f

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X \}$$

Max-weight func f is M$^\natural$-concave (Murota 1996)

- Proof by augmenting path
- Extension to min-cost network flow

Polynomial Matrix

(Dress-Wenzel 90)
Valuated Matroid

$$A = \begin{array}{|c|c|c|c|} \hline s+1 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \omega(J) = \deg \det A[J]$$

$\mathcal{B} = \{J \mid J$ is a base of column vectors$\}$

Grassmann-Plücker $\Rightarrow$ Exchange (M-concave)

For any $J, J' \in \mathcal{B}$, $i \in J \setminus J'$, there exists $j \in J' \setminus J$
s.t. $J - i + j \in \mathcal{B}$, $J' + i - j \in \mathcal{B}$,

$$\omega(J) + \omega(J') \leq \omega(J - i + j) + \omega(J' + i - j)$$

Ex. $J = \{1, 2\}$, $\quad J' = \{3, 4\}$, $\quad i = 1$

$\det A[\{1, 2\}] = \det A[\{3, 4\}] = 1$, $\quad \omega(J) = \omega(J') = 0$

Can take $j = 3$: $\quad J - i + j = \{3, 2\}$, $J' + i - j = \{1, 4\}$

$$\omega(J - i + j) = 1,\ \omega(J' + i - j) = 1$$

Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **M-convex functions**

$\Rightarrow$ Part II

Five Properties (Summary/Preview)

	convex ext.	local opt/ global opt	Legendre biconjug.	separat. theorem	Fenchel duality
submod. (set fn)	Y	Y	Y	Y	Y
separable -convex	Y	Y	Y	Y	Y
integrally -convex	Y	Y	Y	N	N
L-convex ($\mathbb{Z}^n$)	Y	Y	Y	Y	Y
M-convex ($\mathbb{Z}^n$)	Y	Y	Y	Y	Y

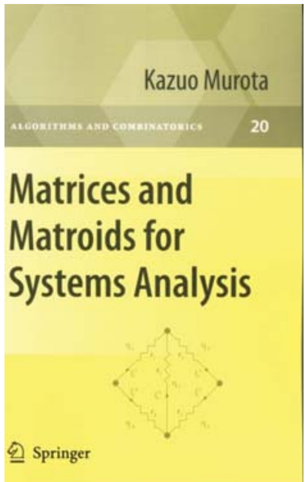
Books (discrete convex analysis)

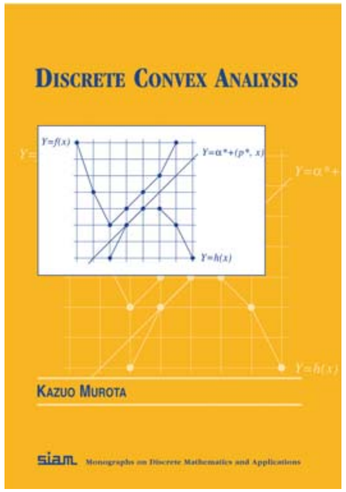
2000: Murota, Matrices and Matroids for Systems Analysis, Springer

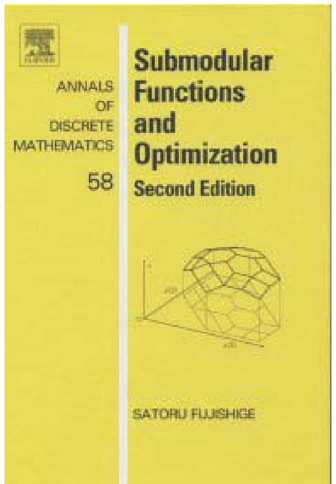
2003: Murota, Discrete Convex Analysis, SIAM

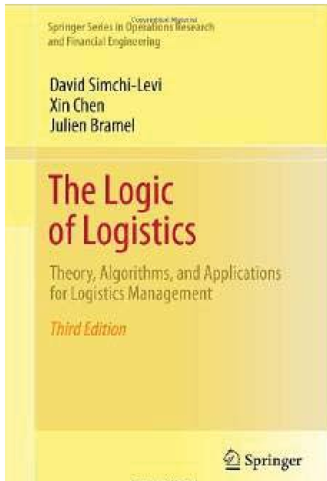
2005: Fujishige, Submodular Functions and Optimization, 2nd ed., Elsevier

2014: Simchi-Levi, Chen, Bramel, The Logic of Logistics, 3rd ed., Springer









離散凸解析 の 参考書

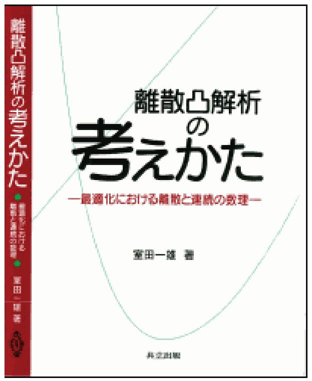
2001: 室田，離散凸解析，共立出版

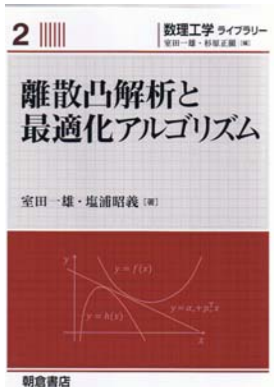
2007: 室田，離散凸解析の考えかた，共立出版

2009: 田村，離散凸解析とゲーム理論，朝倉書店

2013: 室田，塩浦，離散凸解析と最適化アルゴリズム，朝倉

2015: 穴井，斉藤，今日から使える！組合せ最適化，講談社





Survey/Slide/Video/Software

[Survey]

Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

Murota: Discrete convex analysis: A tool for economics and game theory. Journal of Mechanism and Institution Design 1, 151–273 (2016)

[Slide]

http://www.comp.tmu.ac.jp/kzmurota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)

https://ist.ksc.kwansei.ac.jp/~tutimura/DCP/

END