Chapter 1
Main Features of Discrete Convex Analysis

(COSS 2018 Reading Material)

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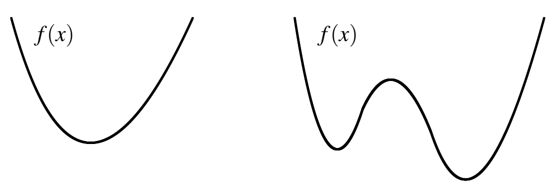
1.1 Aim of Discrete Convex Analysis

Convex functions appear in many mathematical models in engineering, operations research, economics, game theory, and other sciences. The concept of convex functions is explained most easily by examples. The function in Fig. 1.1 (a) is convex and that in Fig. 1.1 (b) is not.



(a) Convex function (b) Nonconvex function

Fig. 1.1 Convex and nonconvex functions

A formal definition of convex functions is as follows. We denote the set of real numbers by $\mathbb{R}$ and let $\overline{\mathbb{R}}$ denote the set $\mathbb{R} \cup \{+\infty\}$. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *convex* if

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \tag{1.1}$$

holds for all $x, y \in \mathbb{R}^n$ and for all λ with $0 \leq \lambda \leq 1$, where it is understood that this inequality is satisfied if $f(x)$ or $f(y)$ is equal to $+\infty$. Figure 1.2 illustrates this inequality; the point ⦿ correponds to the left-hand side of (1.1) and ● to the right-hand side. Alternatively, we can say that a function f is convex if and only if, for any x and y, the line segment connecting $(x, f(x))$ and $(y, f(y))$ lies above the graph of f.

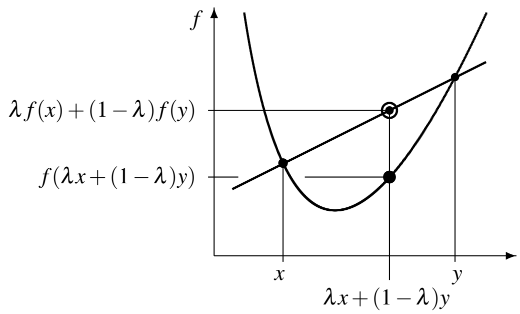


Fig. 1.2 Definition of convex functions

Figure 1.3 illustrates the failure of inequality (1.1) for the function in Fig. 1.1 (b). Again the point ⦿ correponds to the left-hand side of (1.1) and ● to the right-hand side.

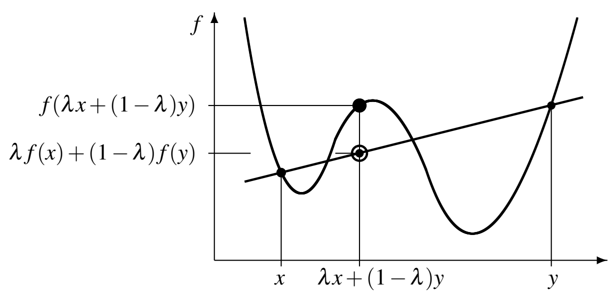


Fig. 1.3 Failure of the convexity condition

Convex functions are amenable to minimization. An obvious reason for this is that a local minimizer of a convex function is guaranteed to be a global minimizer. To put it more precisely, let $x \in \mathbb{R}^n$ be a point at which f is finite. If the inequality $f(x) \leq f(y)$ holds for every y in some neighborhood of x, then the same inequality holds for all $y \in \mathbb{R}^n$. This property enables us to design descent-type algorithms for computing the minimum of a convex function. It should be clear that, for nonconvex functions, local minimality does not imply global minimality. In Fig. 1.4 (b), for example, there are two local minimizers (●, ⦿) of which ● is a global minimizer and ⦿ is not. On the other hand, the convex function in Fig. 1.4 (a) has only one local minimizer, which is also a global minimizer.

Convex analysis lays the foundation for theoretical and algorithmic treatment of convex functions. Besides the equivalence of local and global minimalities men-

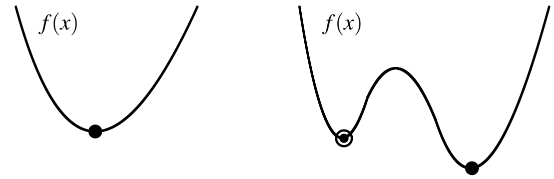


Fig. 1.4 Local and global minimizers of convex and nonconvex functions

tioned above, there are much deeper reasons why convex functions are tractable in optimization (minimization), namely, duality phenomena such as conjugacy, biconjugacy, min-max relations, and separation theorems. Indeed, duality is one of the central issues in convex analysis. Convex analysis is also indispensable in dealing with nonconvex functions.

Discrete convex analysis is aimed at providing an analogous theoretical framework for discrete functions through a combination of convex analysis and the theory of networks and matroids in discrete optimization. Primarily, it is a theory of functions $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ in discrete variables that enjoy certain nice properties comparable to convexity. At the same time, it is a theory of convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ in continuous variables that have additional combinatorial properties.

In defining convexity concepts for functions in discrete variables, it would be natural to expect the following properties:

1. Local minimality guarantees global minimality.
2. Duality theorems such as conjugacy, biconjugacy, min-max relations, and separation theorems hold.

In discrete convex analysis, two convexity concepts, called L-convexity and M-convexity are defined, where “L” stands for “Lattice” and “M” for “Matroid.” L-convex functions and M-convex functions are endowed with the above nice properties and they are conjugate to each other through the (continuous or discrete) Legendre–Fenchel transformation.

The objective of this chapter is to present an overall picture on the most important concepts and properties of discrete convex functions. In Section 1.2 we identity major issues to be addressed in discrete convex analysis by considering discrete convex functions in one variable. Definitions and properties of multivariate discrete convex functions are outlined in Sections 1.3 and 1.4.

Although discrete convex analysis is inspired by concepts and results in convex analysis and combinatorial optimization, familiarity with these areas is not necessary in reading this book. Elementary facts in convex analysis used in this book are presented in Chapter 2, while the interested reader is referred to [2,3,4,27,36,65] for more details of convex analysis. Concerning combinatorial optimization, the reader is referred to [64, 72] for matroids, [5, 13, 35, 66] for combinatorial optimization, and [16, 71] for submodular function theory.

1.2 Univariate Discrete Convex Functions

In this section we investigate discrete convexity for univariate functions (functions in one variable). Univariate discrete convex functions are easy to analyze and useful as a prototype of discrete convex functions. In so doing we intend to identify the general issues to be addressed in discrete convex analysis for multivariate functions.

1.2.1 Definition

We denote the set of all real numbers by $\mathbb{R}$ and the set of all integers by $\mathbb{Z}$. We also use notations $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$, and $\underline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\}$.

We consider functions f that are defined on integers $\mathbb{Z}$ and take values in $\mathbb{R} \cup \{-\infty, +\infty\}$. The effective domain of f, to be denoted as $\mathrm{dom}\, f$, is defined as the set of points at which the value of f is finite:[1]

$$\mathrm{dom}\, f = \mathrm{dom}_{\mathbb{Z}} f = \{x \in \mathbb{Z} \mid -\infty < f(x) < +\infty\}. \tag{1.2}$$

A function $f : \mathbb{Z} \to \overline{\mathbb{R}}$ is said to be a *discrete convex function* (or simply *convex function*) if $\mathrm{dom}\, f \neq \emptyset$ and[2]

$$f(x-1) + f(x+1) \geq 2f(x) \qquad (\forall x \in \mathbb{Z}), \tag{1.3}$$

where it is understood that this inequality is satisfied trivially if $f(x-1) = +\infty$ or $f(x+1) = +\infty$. It follows from (1.3) that the effective domain $\mathrm{dom}\, f$ is a nonempty integer interval (see Remark 1.1), since $f(x-1) < +\infty$ and $f(x+1) < +\infty$ imply $f(x) < +\infty$. The inequality (1.3) can be rewritten as

$$f(x) - f(x-1) \leq f(x+1) - f(x) \qquad (\forall x \in \mathrm{dom}\, f), \tag{1.4}$$

showing the monotonicity (non-decreasingness) of the difference $f(x+1) - f(x)$ on $\mathrm{dom}\, f$. Thus, the discrete convexity of a function f is characterized by the monotonicity of the difference.

Naturally, we say that $g : \mathbb{Z} \to \underline{\mathbb{R}}$ is a *discrete concave function* if $-g : \mathbb{Z} \to \overline{\mathbb{R}}$ is a discrete convex function. That is, $g : \mathbb{Z} \to \underline{\mathbb{R}}$ is a discrete concave function if and only if $\mathrm{dom}\, g \neq \emptyset$ and

$$g(x-1) + g(x+1) \leq 2g(x) \qquad (\forall x \in \mathbb{Z}). \tag{1.5}$$

Remark 1.1. An *integer interval* means the set of integers in an interval. A finite integer interval is a set of the form $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$ for some $a, b \in \mathbb{Z}$. An infinite

[1] We sometimes use $\mathrm{dom}_{\mathbb{Z}} f$ for $\mathrm{dom}\, f$ to emphasize that it is a subset of $\mathbb{Z}$.

[2] Symbol $\forall$ in (1.3) means “for all”, “for any,” or “for each.”

integer interval is a set of the form $\{x \in \mathbb{Z} \mid a \leq x < +\infty\}$, $\{x \in \mathbb{Z} \mid -\infty < x \leq b\}$, or $\{x \in \mathbb{Z} \mid -\infty < x < +\infty\}$ for some $a, b \in \mathbb{Z}$. ∎

In the following sections, we demonstrate that major nice features of convex functions in continuous variables are shared by functions in discrete variables satisfying (1.3).

1.2.2 Convex extension

For any function $f : \mathbb{Z} \to \overline{\mathbb{R}}$ in a single discrete variable, we can associate a piecewise-linear function $\tilde{f} : \mathbb{R} \to \overline{\mathbb{R}}$ in a continuous variable by connecting consecutive points $(x, f(x))$ and $(x+1, f(x+1))$ by line segments for all $x \in \mathbb{Z}$ as in Fig. 1.5. This function $\tilde{f}$ is called the *piecewise-linear extension* of f. It is easy to see that a function f is discrete convex (1.3) if and only if its piecewise-linear extension $\tilde{f}$ is convex (1.1).

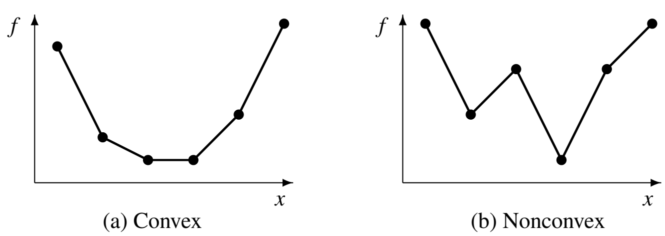


Fig. 1.5 Piecewise-linear extension and convex extensibility

A function $f : \mathbb{Z} \to \overline{\mathbb{R}}$ is said to be *convex-extensible* if there exists a convex function $\overline{f} : \mathbb{R} \to \overline{\mathbb{R}}$ in a continuous variable such that

$$\overline{f}(x) = f(x) \qquad (\forall x \in \mathbb{Z}). \tag{1.6}$$

Such $\overline{f}$ is called a *convex extension* of f. A convex extension may not be unique. Fig. 1.6 shows an example of two different convex extensions for the same function.

A convex-extensible function f is discrete convex, since

$$f(x-1) + f(x+1) = \overline{f}(x-1) + \overline{f}(x+1) \geq 2\overline{f}(x) = 2f(x)$$

by (1.6) and the convexity of $\overline{f}$. Conversely, a discrete convex function f is convex-extensible, since its piecewise-linear extension $\tilde{f}$ serves as a convex extension $\overline{f}$. Thus discrete convexity (1.3) is equivalent to convex-extensibility (1.6), which fact is stated as a theorem below. We emphasize that such a simple characterization of convex-extensibility is valid only for functions in a single variable.

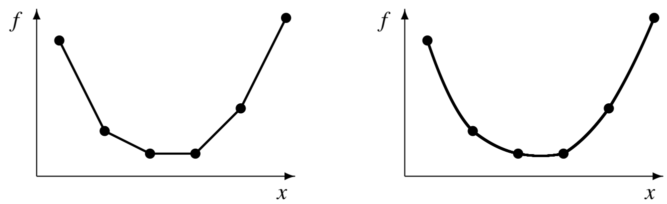


Fig. 1.6 Two different convex extensions of the same function

Theorem 1.1. *A univariate function* $f : \mathbb{Z} \to \overline{\mathbb{R}}$ *is convex-extensible if and only if it satisfies* (1.3). ∎

1.2.3 Minimization

Probably the most appealing property of a convex function is the equivalence of local minimality and global minimality.

The following is a natural analogue of this equivalence for univariate discrete convex functions, where $x \in \mathbb{Z}$ is said to be a (*global*) *minimizer* of $f : \mathbb{Z} \to \overline{\mathbb{R}}$ if $f(x) \leq f(y)$ for all $y \in \mathbb{Z}$.

Theorem 1.2. *For a univariate discrete convex function* $f : \mathbb{Z} \to \overline{\mathbb{R}}$*, an integer* $x \in \mathrm{dom}\, f$ *is a global minimizer of* f *if and only if it is a local minimizer in the sense that*

$$f(x) \leq \min\{f(x-1), f(x+1)\}. \tag{1.7}$$

Proof. The "only-if" part is obvious. Although the "if" part is also easy to see, a formal proof is as follows. Since $f(x+1) - f(x) \geq 0$ by (1.7), it follows from the monotonicity (1.4) of the difference that $f(x+k+1) - f(x+k) \geq 0$ for $k = 0, 1, 2, \ldots$. Hence, if $y > x$, we have $f(y) \geq f(y-1) \geq \cdots \geq f(x+1) \geq f(x)$. Similarly, if $y < x$, we have $f(y) \geq f(y+1) \geq \cdots \geq f(x-1) \geq f(x)$, since $f(x-k-1) - f(x-k) \geq f(x-1) - f(x) \geq 0$ for $k = 0, 1, 2, \ldots$ by (1.4) and (1.7). Therefore, $f(y) \geq f(x)$ for all $y \in \mathbb{Z}$. □

The local characterization of global minimality naturally leads to descent-type algorithms for minimization.

Remark 1.2. An alternative proof of Theorem 1.2 could be obtained by considering the piecewise-linear extension $\tilde{f}$ of f, as in Fig. 1.5 (a), and applying to $\tilde{f}$ the local characterization of global minimality for convex functions in continuous variables. In this section, however, we intentionally avoid using convex extensions to prove theorems for univariate discrete convex functions. ∎

1.2.4 Conjugacy

For any function $f : \mathbb{R} \to \overline{\mathbb{R}}$ in a continuous variable, whether convex or not, the function $f^\bullet : \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$f^\bullet(p) = \sup\{px - f(x) \mid x \in \mathbb{R}\} \qquad (p \in \mathbb{R}) \tag{1.8}$$

is called the (convex) *conjugate* of f, where $\mathrm{dom}\, f \neq \emptyset$ is assumed. For a geometric interpretation, consider the graph $y = f(x)$ and the tangent line with slope p (see Fig. 1.7). Then $-f^\bullet(p)$ is equal to the y-intercept of this tangent line.

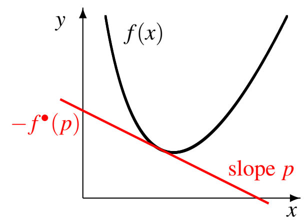


Fig. 1.7 Geometric meaning of the Legendre–Fenchel transformation

The function $f^\bullet$ is also referred to as the (convex) *Legendre–Fenchel transform* (or simply *Legendre transform*) of f, and the mapping $f \mapsto f^\bullet$ as the (convex) *Legendre–Fenchel transformation* (or simply *Legendre transformation*). The conjugate function of the conjugate of f, i.e., $(f^\bullet)^\bullet$, is called the *biconjugate* of f and denoted as $f^{\bullet\bullet}$. See Section 2.4 for more about the Legendre–Fenchel transformation.

It is known in convex analysis[3] that for any function f, the conjugate $f^\bullet$ is a convex function, and the biconjugate $f^{\bullet\bullet}$ is (essentially) the largest convex function that is dominated pointwise by f. In particular, for a convex function f, the biconjugate $f^{\bullet\bullet}$ coincides with f itself (under some regularity assumption). Therefore, the Legendre–Fenchel transformation establishes a symmetric (or involutive) one-to-one correspondence within the class of all (univariate) convex functions.

For a function $f : \mathbb{Z} \to \overline{\mathbb{R}}$ in an integer variable, the discrete version of the Legendre–Fenchel transformation can be defined as

$$f^\bullet(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{R}); \tag{1.9}$$

recall that $\mathrm{dom}_{\mathbb{Z}} f \neq \emptyset$ is assumed. The function $f^\bullet : \mathbb{R} \to \overline{\mathbb{R}}$ is called the (convex) *conjugate* of f. For any function $f : \mathbb{Z} \to \overline{\mathbb{R}}$, which may or may not satisfy (1.3), the conjugate function $f^\bullet : \mathbb{R} \to \overline{\mathbb{R}}$ is a convex function. We call (1.9) the (convex) *discrete Legendre–Fenchel transformation*.

For an integer-valued function f on $\mathbb{Z}$, the value of $f^\bullet(p)$ for integral p is an integer, since both px and $f(x)$ are integers in (1.9). Hence (1.9) with $p \in \mathbb{Z}$ defines

[3] See Theorem 2.2 in Section 2.4 for precise statements involving the condition of "closedness."

a transformation of $f : \mathbb{Z} \to \overline{\mathbb{Z}}$ to $f^{\bullet} : \mathbb{Z} \to \overline{\mathbb{Z}}$. For later reference, (1.9) with $p \in \mathbb{Z}$ is explicitly written here:

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z}). \tag{1.10}$$

This function $f^{\bullet}$ is referred to as the *integral conjugate* of f, and the mapping $f \mapsto f^{\bullet}$ of (1.10) as the *fully-discrete* or *fully-integral* Legendre–Fenchel transformation. We can apply (1.10) twice to obtain $f^{\bullet\bullet} = (f^{\bullet})^{\bullet}$, which is referred to as the *integral biconjugate* of f.

The conjugacy theorem for univariate discrete convex functions reads as follows.

Theorem 1.3. *For an integer-valued univariate discrete convex function $f : \mathbb{Z} \to \overline{\mathbb{Z}}$, the integral conjugate $f^{\bullet}$ in* (1.10) *is another integer-valued univariate discrete convex function. Furthermore, the integral biconjugate $f^{\bullet\bullet}$ coincides with f itself, i.e., $f^{\bullet\bullet} = f$.*

Proof. The proof consists of three parts.

[Discrete convexity (1.3) of $f^{\bullet}$]: The addition of the two expressions

$$f^{\bullet}(p-1) = \sup_{x}\{(p-1)x - f(x)\}, \quad f^{\bullet}(p+1) = \sup_{x}\{(p+1)x - f(x)\}$$

yields $f^{\bullet}(p-1) + f^{\bullet}(p+1) \geq \sup_x\{\big((p+1)x - f(x)\big) + \big((p-1)x - f(x)\big)\} = 2\sup_x\{px - f(x)\} = 2f^{\bullet}(p)$.

[$f^{\bullet\bullet} \leq f$]: For any $x, p \in \mathbb{Z}$ we have $f^{\bullet}(p) \geq px - f(x)$ by (1.10), and hence $px - f^{\bullet}(p) \leq f(x)$. Therefore, $f^{\bullet\bullet}(x) = \sup_p\{px - f^{\bullet}(p)\} \leq f(x)$.

[$f^{\bullet\bullet} \geq f$]: First we assume $x \in \mathrm{dom}\, f$. Take an integer p satisfying $f(x) - f(x-1) \leq p \leq f(x+1) - f(x)$, which is possible by (1.4) and the integrality of the function value. Consider the function $h(y) = f(y) - py$ in $y \in \mathbb{Z}$. Then we have $h(x) \leq \min\{h(x-1), h(x+1)\}$. This implies, by Theorem 1.2, that $h(x) \leq h(y)$ for all $y \in \mathbb{Z}$, which is equivalent to $px - f(x) \geq py - f(y)$ $(y \in \mathbb{Z})$. Hence we have $px - f(x) = \sup_y\{py - f(y)\} = f^{\bullet}(p)$. Therefore, $f^{\bullet\bullet}(x) = \sup_q\{qx - f^{\bullet}(q)\} \geq px - f^{\bullet}(p) = f(x)$.

Next we consider the case of $x \notin \mathrm{dom}\, f$. We assume that $\mathrm{dom}\, f$ is an integer interval $[a,b]_{\mathbb{Z}}$ with b finite and $x \geq b+1$; the other case with a finite and $x \leq a-1$ can be treated similarly. For all sufficiently large $p \in \mathbb{Z}$, say $p \geq p_0$, we have $f^{\bullet}(p) = pb - f(b)$. Therefore, $f^{\bullet\bullet}(x) = \sup_{p\in\mathbb{Z}}\{px - f^{\bullet}(p)\} \geq \sup_{p \geq p_0}\{px - f^{\bullet}(p)\} = \sup_{p \geq p_0}\{p(x-b) + f(b)\} = +\infty$. □

Theorem 1.3 above shows that the fully-integral Legendre–Fenchel transformation (1.10) establishes a symmetric (or involutive) one-to-one correspondence within the class of all integer-valued univariate discrete convex functions.

Example 1.1. The Legendre–Fenchel transformation is illustrated for a quadratic function $f(x) = x^2$ in continuous and discrete variables. In the continuous case with $x \in \mathbb{R}$, we have

$$f^{\bullet}(p) = \sup\{px - x^2 \mid x \in \mathbb{R}\} = \frac{1}{4}p^2 \qquad (p \in \mathbb{R})$$

by (1.8). In the discrete case with $x \in \mathbb{Z}$, the discrete Legendre–Fenchel transformation (1.9) gives

$$\begin{aligned} f^{\bullet}(p) &= \sup\{px - x^2 \mid x \in \mathbb{Z}\} \\ &= \max\{px - x^2 \mid x \in \{\lfloor p/2 \rfloor, \lceil p/2 \rceil\}\} \\ &= \max\left\{ \left\lfloor \frac{p}{2} \right\rfloor \left(p - \left\lfloor \frac{p}{2} \right\rfloor\right), \left\lceil \frac{p}{2} \right\rceil \left(p - \left\lceil \frac{p}{2} \right\rceil\right) \right\} \qquad (p \in \mathbb{R}). \end{aligned}$$

This is a piecewise-linear convex function whose graph consists of line segments connecting $(2k-1, k^2-k)$ and $(2k+1, k^2+k)$ for $k \in \mathbb{Z}$. In the fully discrete case with $x \in \mathbb{Z}$ and $p \in \mathbb{Z}$, we obtain the integral conjugate

$$f^{\bullet}(p) = \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil \qquad (p \in \mathbb{Z}),$$

since $\left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{p}{2} \right\rceil = p$ for an integer p. The integral biconjugacy $f^{\bullet\bullet} = f$ stated in Theorem 1.3 is given as the identity

$$\sup\left\{ px - \left\lfloor \frac{p}{2} \right\rfloor \cdot \left\lceil \frac{p}{2} \right\rceil \mid p \in \mathbb{Z} \right\} = x^2 \qquad (x \in \mathbb{Z}),$$

which can be verified easily. ∎

For a general (not necessarily discrete convex) function f we have the following statement about $f^{\bullet}$ and $f^{\bullet\bullet}$.

Proposition 1.1. *For any integer-valued univariate function $f : \mathbb{Z} \to \overline{\mathbb{Z}}$, the integral conjugate $f^{\bullet}$ in* (1.10) *is an integer-valued discrete convex function, and the integral biconjugate $f^{\bullet\bullet}$ is the largest integer-valued discrete convex function such that $f^{\bullet\bullet}(x) \le f(x)$ for all $x \in \mathbb{Z}$.*

Proof. In view of the proof of Theorem 1.3 it remains to prove that $f^{\bullet\bullet}$ is the largest such function. Let $g : \mathbb{Z} \to \overline{\mathbb{Z}}$ be any integer-valued univariate discrete convex function satisfying $g(x) \le f(x)$ for all $x \in \mathbb{Z}$. By (1.10) we have $g^{\bullet}(p) \ge f^{\bullet}(p)$ for all $p \in \mathbb{Z}$. This implies, again by (1.10), that $g^{\bullet\bullet}(x) \le f^{\bullet\bullet}(x)$ for all $x \in \mathbb{Z}$. Here we have $g^{\bullet\bullet}(x) = g(x)$ by Theorem 1.3. Therefore, $g(x) \le f^{\bullet\bullet}(x)$ for all $x \in \mathbb{Z}$. □

Example 1.2. Proposition 1.1 is illustrated for a simple nonconvex function. Let f be defined by $f(0) = 0$, $f(1) = f(2) = 3$, and $f(x) = +\infty$ for $x \notin \{0,1,2\}$. A direct calculation using the fully-integral Legendre–Fenchel transformation (1.10) shows

$$f^{\bullet}(p) = \max\{px - f(x) \mid x \in \{0,1,2\}\} = \max\{0, 2p-3\} \qquad (p \in \mathbb{Z}),$$

$$f^{\bullet\bullet}(x) = \sup\{px - \max\{0, 2p-3\} \mid p \in \mathbb{Z}\} = \begin{cases} 0 & (x = 0), \\ 1 & (x = 1), \\ 3 & (x = 2), \\ +\infty & (x \in \mathbb{Z} \setminus \{0,1,2\}). \end{cases}$$

This function $f^{\bullet\bullet}$ is indeed the largest integer-valued discrete convex function satisfying $f^{\bullet\bullet}(x) \leq f(x)$ for all $x \in \mathbb{Z}$. However, if no integrality is imposed on the function value, the largest discrete convex function $g : \mathbb{Z} \to \overline{\mathbb{R}}$ satisfying $g(x) \leq f(x)$ for all $x \in \mathbb{Z}$ is given by: $g(0) = 0$, $g(1) = 3/2$, $g(2) = 3$, and $g(x) = +\infty$ for $x \notin \{0, 1, 2\}$. ∎

1.2.5 Discrete separation theorem

The separation theorem for functions in a continuous variable asserts the existence of an affine function that lies between a convex function and a concave function (Fig. 1.8). While precise technical conditions are specified in Theorem 2.3 in Section 2.5, the separation theorem can be stated roughly as follows.

Theorem 1.4. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a convex function and $g : \mathbb{R} \to \underline{\mathbb{R}}$ be a concave function (satisfying certain regularity conditions). If $f(x) \geq g(x)$ for all $x \in \mathbb{R}$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}$ such that*

$$f(x) \geq \alpha^* + p^* x \geq g(x) \qquad (\forall x \in \mathbb{R}). \tag{1.11}$$

∎

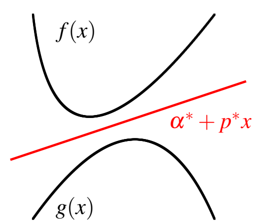
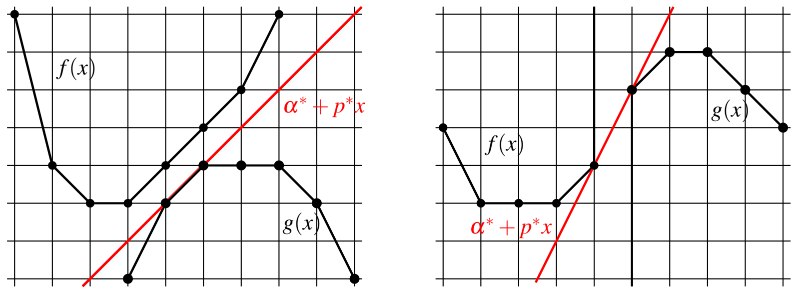


Fig. 1.8 Separation theorem for functions in a continuous variable

For discrete convex and concave functions we have the following *discrete separation theorem*, which is a discrete counterpart of Theorem 1.4. Discreteness is incorporated twofold in the theorem. The first statement of Theorem 1.5 is concerned with discreteness in the variable in that "$x \in \mathbb{R}$" in Theorem 1.4 is changed to "$x \in \mathbb{Z}$." The second statement is concerned with discreteness in function values in that the separating affine function is described by integral α^* and p^* for integer-valued functions f and g. Figure 1.9 illustrates the integer-valued case.

Theorem 1.5. *Let $f : \mathbb{Z} \to \overline{\mathbb{R}}$ be a discrete convex function and $g : \mathbb{Z} \to \underline{\mathbb{R}}$ be a discrete concave function. If $f(x) \geq g(x)$ for all $x \in \mathbb{Z}$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}$ such that*

$$f(x) \geq \alpha^* + p^* x \geq g(x) \qquad (\forall x \in \mathbb{Z}). \tag{1.12}$$



(a) $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ (b) $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$

Fig. 1.9 Discrete separation theorem

Moreover, if f and g are integer-valued, there exist integral $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}$.

Proof. (The claim will be intuitively obvious from Fig. 1.9, but we give a formal detailed proof for interested readers.) Assume that $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g \neq \emptyset$; the case with $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g = \emptyset$ is treated at the end of the proof. Define $h(x) = f(x) - g(x)$ for $x \in \mathbb{Z}$ and $\Delta = \inf\{h(x) \mid x \in \mathbb{Z}\}$. Note that h is a discrete convex function and Δ is finite and nonnegative by the assumption.

First suppose that the infimum Δ is attained and let z be a minimizer of h. Let $F^+ = f(z+1) - f(z)$, $F^- = f(z) - f(z-1)$, $G^+ = g(z+1) - g(z)$, and $G^- = g(z) - g(z-1)$. We have $F^+ \geq G^+$ and $F^- \leq G^-$ by $h(z+1) \geq h(z)$ and $h(z-1) \geq h(z)$, respectively, whereas $F^- \leq F^+$ and $G^- \geq G^+$ by (1.4). Hence $\max(F^-, G^+) \leq \min(F^+, G^-)$, which implies that there exists $p^* \in \mathbb{R}$ satisfying[4]

$$\max(F^-, G^+) \leq p^* \leq \min(F^+, G^-). \tag{1.13}$$

Let $\alpha^* = f(z) - p^* z$. Then $g(z) - p^* z \leq f(z) - p^* z = \alpha^*$. Since $F^- \leq p^* \leq F^+$, we have $f(y) - f(z) \geq p^*(y - z)$ for all $y \in \mathbb{Z}$, i.e., $f(y) - p^* y \geq f(z) - p^* z = \alpha^*$ for all $y \in \mathbb{Z}$. Similarly, since $G^+ \leq p^* \leq G^-$, we have $g(y) - g(z) \leq p^*(y - z)$ for all $y \in \mathbb{Z}$, i.e., $g(y) - p^* y \leq g(z) - p^* z \leq f(z) - p^* z = \alpha^*$ for all $y \in \mathbb{Z}$. Therefore, we obtain $f(y) - p^* y \geq \alpha^* \geq g(y) - p^* y$ for all $y \in \mathbb{Z}$, which is equivalent to (1.12).

Next suppose that the infimum Δ is not attained.[5] Then either $\lim_{x \to -\infty} h(x) = \Delta$ or $\lim_{x \to +\infty} h(x) = \Delta$. We consider the latter case only, as the former case can be treated similarly. Then we have $\lim_{x \to +\infty} (h(x+1) - h(x)) = 0$. Since $h(x+1) - h(x) = (f(x+1) - f(x)) - (g(x+1) - g(x))$ with $f(x+1) - f(x)$ nondecreasing and $g(x+1) - g(x)$ nonincreasing, we have

$$\lim_{x \to +\infty} (f(x+1) - f(x)) = \lim_{x \to +\infty} (g(x+1) - g(x)) = p^*$$

---

[4] Using the notation of subgradients (Section 16.6), the condition (1.13) can be expressed as $p^* \in \partial f(z) \cap \partial(-g)(z)$.

[5] For example, such case occurs for $f(x) = x + 2 + \exp(-x)$ and $g(x) = x - \exp(-2x)$. The infimum Δ is equal to 2, but it is not attained. The proof yields $p^* = 1$ and $\alpha^* = 2$.

for some $p^* \in \mathbb{R}$. Let $\alpha^* = \lim_{y \to +\infty}(f(y) - p^* y)$. Since $f(x) - p^* x$ is nonincreasing in x, we obtain $f(x) - p^* x \geq \alpha^*$ for all $x \in \mathbb{Z}$. Similarly, since $g(x) - p^* x$ is nondecreasing in x, we obtain $g(x) - p^* x \leq \lim_{y \to +\infty}(g(y) - p^* y) \leq \lim_{y \to +\infty}(f(y) - p^* y) = \alpha^*$ for all $x \in \mathbb{Z}$. Therefore, (1.12) holds.

For integer-valued f and g, the infimum of $h = f - g$ is always attained and F^-, F^+, G^+, and G^- are integers. Hence we can choose an integral p^* in (1.13), and then $\alpha^* = f(z) - p^* z$ is also integral.

It remains to treat the case of $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g = \emptyset$. We only consider the case where $\mathrm{dom}_{\mathbb{Z}} f = [a,b]_{\mathbb{Z}}$ and $\mathrm{dom}_{\mathbb{Z}} g = [c,d]_{\mathbb{Z}}$ with $b+1 \leq c$. The inequality (1.12) holds if $p^* \geq \max\{f(b) - f(b-1), g(c+1) - g(c), (g(c) - f(b))/(c-b)\}$ and $\alpha^* = f(b) - p^* b$. For integer-valued f and g, we can choose integral p^* and α^*. □

As we have seen above, the discrete separation theorem for univariate discrete convex functions is a fairly simple statement that can be proved without much difficulty. For multivariate discrete convex functions, however, the discrete separation theorem is highly nontrivial and often captures deep combinatorial properties in spite of its apparent similarity to the separation theorem in convex analysis.

1.2.6 Fenchel-type duality theorem

For integer-valued discrete convex and concave functions $f : \mathbb{Z} \to \overline{\mathbb{Z}}$ and $g : \mathbb{Z} \to \underline{\mathbb{Z}}$, we consider the problem of minimizing $f(x) - g(x)$ over $x \in \mathbb{Z}$. For this problem we have the following Fenchel-type min-max duality theorem. We define the concave version of the fully-integral Legendre–Fenchel transformation as

$$g^{\circ}(p) = \inf\{px - g(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z}). \tag{1.14}$$

We have $g^{\circ\circ} = g$ (integral biconjugacy) by Theorem 1.3, where $g^{\circ\circ}$ denotes $(g^{\circ})^{\circ}$.

Theorem 1.6. *Let $f : \mathbb{Z} \to \overline{\mathbb{Z}}$ be an integer-valued discrete convex function and $g : \mathbb{Z} \to \underline{\mathbb{Z}}$ be an integer-valued discrete concave function such that* $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbb{Z}} g^{\circ} \neq \emptyset$. *Then we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbb{Z}\} = \sup\{g^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbb{Z}\}. \tag{1.15}$$

If this common value is finite, the infimum and the supremum are attained.

Proof. Suppose that $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g \neq \emptyset$. For any $x, p \in \mathbb{Z}$ we have $f^{\bullet}(p) \geq px - f(x)$ by (1.10) and $g^{\circ}(p) \leq px - g(x)$ by (1.14), and therefore, $g^{\circ}(p) - f^{\bullet}(p) \leq f(x) - g(x)$. This shows $\inf(f - g) \geq \sup(g^{\circ} - f^{\bullet})$ (weak duality).

Let $\Delta = \inf\{f(x) - g(x) \mid x \in \mathbb{Z}\}$, which is an integer or $-\infty$. If $\Delta = -\infty$, the weak duality implies $\sup(g^{\circ} - f^{\bullet}) = -\infty$ and hence (1.15). Suppose that Δ is finite. Then the infimum is attained since the functions are integer-valued. By the discrete separation theorem (Theorem 1.5) for $(f - \Delta, g)$, there exist $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}$ such that $f(x) - \Delta \geq \alpha^* + p^* x \geq g(x)$ for all $x \in \mathbb{Z}$. This implies

$g^\circ(p^*) - f^\bullet(p^*) \geq (-\alpha^*) - (-\alpha^* - \Delta) = \Delta$. Combined with the weak duality, this shows (1.15) together with the attainment of the supremum by p^*.

The other case with $\mathrm{dom}_{\mathbb{Z}} f^\bullet \cap \mathrm{dom}_{\mathbb{Z}} g^\circ \neq \emptyset$ can be treated similarly by virtue of the integral biconjugacy $f^{\bullet\bullet} = f$ and $g^{\circ\circ} = g$. Indeed, by applying the above argument to $f^\bullet(p)$ and $g^\circ(p)$ we obtain $\inf\{f^\bullet(p) - g^\circ(p) \mid p \in \mathbb{Z}\} = \sup\{g^{\circ\circ}(x) - f^{\bullet\bullet}(x) \mid x \in \mathbb{Z}\}$, which is equivalent to (1.15). □

As the proof indicates, we can regard the Fenchel-type duality theorem as a corollary of the discrete separation theorem. The converse is also true and it may safely be said that these two theorems are essentially equivalent to each other and capture the same duality principle for univariate discrete convex functions.

1.2.7 Toward multivariate functions

In this section we have clarified the issues of our interest by considering univariate "discrete convex" functions. Fortunately, the natural definition of discrete convexity by the inequality $f(x-1) + f(x+1) \geq 2f(x)$ $(\forall x \in \mathbb{Z})$ in (1.3) entails the following nice properties.

1. Convex-extensibility (Theorem 1.1),
2. Local characterization of global minimality (Theorem 1.2),
3. Conjugacy and biconjugacy under the fully-integral Legendre–Fenchel transformation (Theorem 1.3),
4. Discrete separation theorem (Theorem 1.5),
5. Fenchel-type min-max duality (Theorem 1.6).

Also for multivariate functions it would be natural to expect these five properties of "discrete convex" functions and, accordingly, the key question here is what should be the definition of discrete convexity for multivariate functions in integer variables that leads us to the desired properties. Discrete convex analysis answers this question by introducing two classes of discrete convex functions, called L-convex functions and M-convex functions, as well as their variants called L$^\natural$-convex functions and M$^\natural$-convex functions.[6] These classes of discrete convex functions are described briefly in Section 1.3 as a preview.

[6] "L$^\natural$" and "M$^\natural$" should be read "ell natural" and "em natural," respectively.

1.3 Classes of Discrete Convex Functions

In this section we briefly introduce the major classes of multivariate discrete convex functions defined on the integer lattice $\mathbb{Z}^n$, with particular reference to the following five properties possessed by univariate discrete convex functions:

1. Convex-extensibility,
2. Local characterization of global minimality,
3. Conjugacy and biconjugacy for integer-valued functions under the fully-integral Legendre–Fenchel transformation,
4. Discrete separation theorem,
5. Fenchel-type min-max duality for integer-valued functions.

In the following we briefly describe the definitions and properties of separable convex, integrally convex, L-convex, L$^\natural$-convex, M-convex, and M$^\natural$-convex functions. A full-length description of these and other classes of discrete convex functions are given in Chapter 11.

1.3.1 General issues

For $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, the *effective domain* of f is defined as

$$\mathrm{dom}\, f = \mathrm{dom}_{\mathbb{Z}} f = \{x \in \mathbb{Z}^n \mid -\infty < f(x) < +\infty\}. \tag{1.16}$$

We always assume that $\mathrm{dom}\, f$ is nonempty. The set of minimizers of f is denoted as

$$\mathrm{argmin}\, f = \mathrm{argmin}_{\mathbb{Z}} f = \{x \in \mathbb{Z}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbb{Z}^n)\}. \tag{1.17}$$

Note that $\mathrm{argmin}\, f$ can possibly be an empty set.

Some definitions and remarks are in order concerning the above five properties. A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is said to be *convex-extensible* if there exists a convex function $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ in continuous variables such that

$$\overline{f}(x) = f(x) \qquad (\forall x \in \mathbb{Z}^n). \tag{1.18}$$

Such $\overline{f}$ is called a *convex extension* of f.

To formulate a local characterization of global minimality, we need to specify an appropriate neighborhood of a point (vector) in accordance with the class of functions in question. Then we say that a vector x is a local minimizer if it is a minimizer within that neighborhood of x.

Conjugacy and biconjugacy are defined with respect to the Legendre–Fenchel transformation. For a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ we define $f^\bullet : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{R}^n), \tag{1.19}$$

where $\langle p,x \rangle$ denotes the inner product of p and x, i.e., $\langle p,x \rangle = \sum_{i=1}^{n} p_i x_i$ for $p = (p_1, p_2, \ldots, p_n)$ and $x = (x_1, x_2, \ldots, x_n)$. This function $f^{\bullet}$ is referred to as the *conjugate* of f, and the mapping $f \mapsto f^{\bullet}$ of (1.19) as the *discrete Legendre–Fenchel transformation*.

For an integer-valued function $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ we can define $f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ by

$$f^{\bullet}(p) = \sup\{\langle p,x \rangle - f(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{Z}^n). \tag{1.20}$$

This function $f^{\bullet}$ is referred to as the *integral conjugate* of f, and the mapping $f \mapsto f^{\bullet}$ of (1.20) as the *fully-discrete* or *fully-integral* Legendre–Fenchel transformation. We can apply (1.20) twice to obtain $f^{\bullet\bullet} = (f^{\bullet})^{\bullet}$, which is referred to as the *integral biconjugate* of f.

We are concerned with the following questions related to conjugacy and biconjugacy for any integer-valued function f in a given class of discrete convex functions.

- Does the integral conjugate $f^{\bullet}$ in (1.20) belong to the same class? If not, how is it characterized?
- Do we have integral biconjugacy $f^{\bullet\bullet} = f$ under the transformation (1.20)?

The discrete separation theorem may be stated in a generic form as follows, where a precise meaning of f being "discrete convex" should be specified and g being "discrete concave" means $-g$ being "discrete convex" in the same specific sense.

Theorem 1.7 (Generic form of discrete separation theorem). *Let $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ be a "discrete convex" function and $g : \mathbb{Z}^n \to \underline{\mathbb{R}}$ be a "discrete concave" function such that* $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g \neq \emptyset$*. If $f(x) \geq g(x)$ for all $x \in \mathbb{Z}^n$, there exist $\alpha^* \in \mathbb{R}$ and $p^* \in \mathbb{R}^n$ such that*

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq g(x) \qquad (\forall x \in \mathbb{Z}^n). \tag{1.21}$$

Moreover, if f and g are integer-valued, there exist integral $\alpha^* \in \mathbb{Z}$ and $p^* \in \mathbb{Z}^n$. ∎

The generic form of a Fenchel-type duality theorem reads as follows. For $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ and $g : \mathbb{Z}^n \to \underline{\mathbb{Z}}$, their integral conjugates $f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ and $g^{\circ} : \mathbb{Z}^n \to \underline{\mathbb{Z}}$ are defined, respectively, by the fully-integral Legendre–Fenchel transformation (1.20) and its concave version

$$g^{\circ}(p) = \inf\{\langle p,x \rangle - g(x) \mid x \in \mathbb{Z}^n\} \qquad (p \in \mathbb{Z}^n). \tag{1.22}$$

Theorem 1.8 (Generic form of Fenchel-type duality theorem). *Let $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$ be an integer-valued "discrete convex" function and $g : \mathbb{Z}^n \to \underline{\mathbb{Z}}$ be an integer-valued "discrete concave" function such that* $\mathrm{dom}_{\mathbb{Z}} f \cap \mathrm{dom}_{\mathbb{Z}} g \neq \emptyset$ *or* $\mathrm{dom}_{\mathbb{Z}} f^{\bullet} \cap \mathrm{dom}_{\mathbb{Z}} g^{\circ} \neq \emptyset$*. Then we have*

$$\inf\{f(x) - g(x) \mid x \in \mathbb{Z}^n\} = \sup\{g^{\circ}(p) - f^{\bullet}(p) \mid p \in \mathbb{Z}^n\}. \tag{1.23}$$

If this common value is finite, the infimum and the supremum are attained. ∎

Let $N = \{1,2,\ldots,n\}$. For $i \in N$, the ith unit vector is denoted as $\mathbf{1}^i$; in addition we define $\mathbf{1}^i$ with $i = 0$ to be the zero vector and $\mathbf{1}$ to be the all one vector:

$$\mathbf{1}^i = (0,\ldots,0,\overset{\overset{i}{\vee}}{1},0,\ldots,0) \ \ (i \in N), \quad \mathbf{1}^0 = (0,0,\ldots,0), \quad \mathbf{1} = (1,1,\ldots,1).$$

The characteristic vector of a subset $A \subseteq N$ is denoted by $\mathbf{1}^A$, whose ith component $\mathbf{1}^A_i$ is given by

$$\mathbf{1}^A_i = \begin{cases} 1 \ (i \in A), \\ 0 \ (i \in N \setminus A). \end{cases} \tag{1.24}$$

1.3.2 Separable convex functions

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is called a *separable convex function* if it can be represented as

$$f(x) = \sum_{i=1}^{n} \varphi_i(x_i) \tag{1.25}$$

with univariate discrete convex functions $\varphi_i : \mathbb{Z} \to \overline{\mathbb{R}}$ $(i = 1,2,\ldots,n)$, which, by definition (1.3), satisfy

$$\varphi_i(t-1) + \varphi_i(t+1) \geq 2\varphi_i(t) \qquad (\forall t \in \mathbb{Z}). \tag{1.26}$$

It should be clear that x_i denotes the ith component of vector $x = (x_1, x_2, \ldots, x_n)$.

The following properties of separable convex functions can be shown easily from the corresponding statements for univariate discrete convex functions in Section 1.2.

1. A separable convex function is convex-extensible.
2. For a separable convex function f, a point $x \in \mathrm{dom}_{\mathbb{Z}} f$ is a global minimizer of f if and only if it is a local minimizer in the sense that

$$f(x) \leq \min\{f(x - \mathbf{1}^i), f(x + \mathbf{1}^i)\} \qquad (\forall i \in \{1,2,\ldots,n\}). \tag{1.27}$$

3. The integral conjugate $f^{\bullet}$ in (1.20) of an integer-valued separable convex function f is another integer-valued separable convex function. Furthermore, we have integral biconjugacy $f^{\bullet\bullet} = f$ using the fully-integral Legendre–Fenchel transformation (1.20).
4. A discrete separation theorem of the form of Theorem 1.7 holds for separable convex functions.
5. A Fenchel-type min-max duality of the form of Theorem 1.8 holds for integer-valued separable convex functions.

1.3.3 Integrally convex functions

The concept of integrally convex functions arise from convex-extensibility respecting the integer lattice $\mathbb{Z}^n$.

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is called *integrally convex* if its local convex extension $\tilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is (globally) convex in the ordinary sense, where $\tilde{f}$ is defined as the collection of piecewise-linear convex extensions of f in each unit hypercube $\{x \in \mathbb{R}^n \mid a_i \leq x_i \leq a_i + 1 \ (i = 1,2,\ldots,n)\}$ with $a \in \mathbb{Z}^n$; see Section 11.3 for details. In the case of $n = 1$, integral convexity is equivalent to the condition (1.3).

Among the major properties we are interested in, convex-extensibility, local characterization of global minimality, and integral biconjugacy hold for integrally convex functions, whereas the separation and Fenchel-type min-max theorems fail.

1. An integrally convex function is convex-extensible, by definition.
2. For an integrally convex function f, a point $x \in \mathrm{dom}_{\mathbb{Z}} f$ is a global minimizer of f if and only if it is a local minimizer in the sense that

$$f(x) \leq f(x+d) \qquad (\forall d \in \{-1,0,1\}^n). \tag{1.28}$$

3. The integral conjugate $f^{\bullet}$ in (1.20) of an integer-valued integrally convex function f is not necessarily an integrally convex function. Nevertheless, we have integral biconjugacy $f^{\bullet\bullet} = f$ under the fully-integral Legendre–Fenchel transformation (1.20).
4. No discrete separation theorem of the form of Theorem 1.7 holds for integrally convex functions.
5. No Fenchel-type min-max duality of the form of Theorem 1.8 holds for integer-valued integrally convex functions.

It is noteworthy that the local characterization of global minimality above is formulated in terms of a neighborhood that does not depend on individual functions. Convex-extensibility alone would not lead to such a statement that refers to vectors in $(x+S) \cap \mathbb{Z}^n$ with some $S \subseteq \mathbb{R}^n$ independent of individual functions f.

However, the failure of duality theorems for integrally convex functions indicates the need of some additional conditions for functions that can be qualified as "discrete convex" functions in the true sense of the word.

1.3.4 L-convex functions

We introduce the concept of L-convex functions by featuring an equivalent variant thereof, called L$^\natural$-convex functions ("L" stands for "Lattice" and "L$^\natural$" should be read "ell natural").

We first observe that a convex function f on $\mathbb{R}^n$ satisfies

$$f(x)+f(y) \geq f\left(\frac{x+y}{2}\right)+f\left(\frac{x+y}{2}\right) \qquad (x,y \in \mathbb{R}^n), \tag{1.29}$$

which is a special case of (1.1) with $\lambda = 1/2$. This property, called *midpoint convexity*, is known to be equivalent to convexity if f is a continuous function.

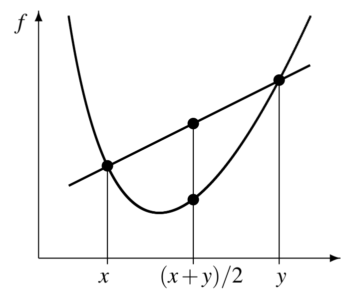


Fig. 1.10 Midpoint convexity

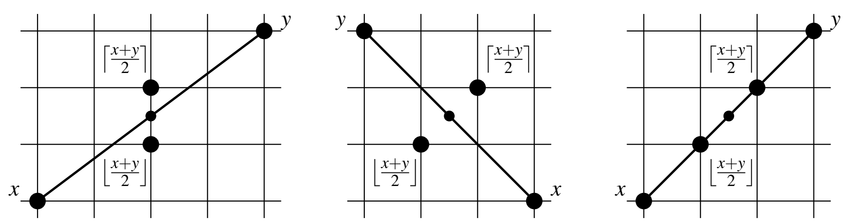


Fig. 1.11 Discrete midpoint convexity

For a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ in discrete variables the above inequality does not always make sense, since the midpoint $(x+y)/2$ of two integer vectors x and y may not be integral. Instead we simulate (1.29) by

$$f(x)+f(y) \geq f\left(\left\lceil\frac{x+y}{2}\right\rceil\right)+f\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right) \qquad (x,y \in \mathbb{Z}^n), \tag{1.30}$$

where, for $z \in \mathbb{R}$ in general, $\lceil z \rceil$ denotes the smallest integer not smaller than z (rounding-up to the nearest integer) and $\lfloor z \rfloor$ the largest integer not larger than z (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications. This is illustrated in Fig. 1.11 in the case of $n = 2$. We refer to (1.30) as *discrete midpoint convexity*.

We say that a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is *L$^\natural$-convex* if it satisfies discrete midpoint convexity (1.30). In the case of $n = 1$, L$^\natural$-convexity is equivalent to the condition (1.3).

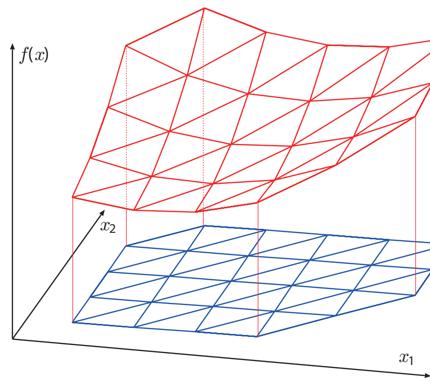


Fig. 1.12 An L$^\natural$-convex function ($n = 2$)

L$^\natural$-convexity is closely related to submodularity. For two vectors x and y, the vectors of componentwise maximum and minimum are denoted respectively by $x \vee y$ and $x \wedge y$, that is,

$$(x \vee y)_i = \max(x_i, y_i), \qquad (x \wedge y)_i = \min(x_i, y_i). \tag{1.31}$$

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is called *submodular* if

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathbb{Z}^n), \tag{1.32}$$

and *translation submodular* if

$$f(x) + f(y) \geq f((x - \alpha\mathbf{1}) \vee y) + f(x \wedge (y + \alpha\mathbf{1})) \quad (\alpha \in \mathbb{Z}_+,\ x, y \in \mathbb{Z}^n), \tag{1.33}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbb{Z}_+$ denotes the set of nonnegative integers. Note that submodularity (1.32) is a special case of translation submodularity (1.33) for $\alpha = 0$.

Translation submodularity characterizes L$^\natural$-convexity. That is, a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is discrete midpoint convex (1.30) if and only if it is translation submodular (1.33). Note that in case of $n = 1$, every function satisfies (1.32), which shows that submodularity alone does not imply discrete convexity. It is known that submodular integrally convex functions are precisely L$^\natural$-convex functions.

For L$^\natural$-convex functions the five key properties take the following forms. It is noteworthy that the conjugacy statement involves another kind of discrete convexity called M$^\natural$-convexity, to be introduced in Section 1.3.5.

1. An L$^\natural$-convex function is convex-extensible.

2. For an $\mathrm{L}^\natural$-convex function f, a point $x \in \mathrm{dom}_{\mathbb{Z}} f$ is a global minimizer of f if and only if it is a local minimizer in the sense that

$$f(x) \leq \min\{f(x-d), f(x+d)\} \qquad (\forall d \in \{0,1\}^n). \tag{1.34}$$

3. The integral conjugate $f^\bullet$ in (1.20) of an integer-valued $\mathrm{L}^\natural$-convex function f is an integer-valued $\mathrm{M}^\natural$-convex function. Furthermore, we have integral biconjugacy $f^{\bullet\bullet} = f$ under the fully-integral Legendre–Fenchel transformation.
4. A discrete separation theorem of the form of Theorem 1.7 holds for $\mathrm{L}^\natural$-convex functions.
5. A Fenchel-type min-max duality of the form of Theorem 1.8 holds for integer-valued $\mathrm{L}^\natural$-convex functions.

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is said to be an *L-convex function* if it is an $\mathrm{L}^\natural$-convex function with the additional property that f is linear in the direction of $\mathbf{1}$, i.e.,

$$f(x+\mathbf{1}) = f(x) + r \qquad (x \in \mathbb{Z}^n) \tag{1.35}$$

for some $r \in \mathbb{R}$ (r being independent of x). It is known that f is L-convex if and only if it satisfies (1.32) and (1.35). L-convex functions and $\mathrm{L}^\natural$-convex functions are essentially equivalent in the sense that $\mathrm{L}^\natural$-convex functions in n variables can be identified, up to the constant r in (1.35), with L-convex functions in $n+1$ variables.

L-convex functions also enjoy the five key properties in slightly modified forms. For an L-convex function f with $r = 0$, the local minimality condition (1.34) is changed to

$$f(x) \leq f(x+d) \qquad (\forall d \in \{0,1\}^n), \tag{1.36}$$

and the conjugate of an L-convex function is an M-convex function, to be introduced in Section 1.3.5.

1.3.5 M-convex functions

Just as L-convexity is defined through discretization of midpoint convexity, another kind of discrete convexity, called M-convexity, can be defined through discretization of another property of convex functions in continuous variables. We feature a variant of M-convexity, called $\mathrm{M}^\natural$-convexity ("M" stands for "Matroid" and "$\mathrm{M}^\natural$" should be read "em natural").

We first observe that a convex function f on $\mathbb{R}^n$ satisfies the inequality

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y)) \tag{1.37}$$

for every $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$. This inequality follows from (1.1) by adding the inequality for $\lambda = \alpha$ and that for $\lambda = 1 - \alpha$. The inequality (1.37) says that the sum of the function values evaluated at two points, x and y, does not increase if the two points approach each other by the same distance on the line segment connecting them (see Fig. 1.13). We refer to this property as *equidistance convexity*.

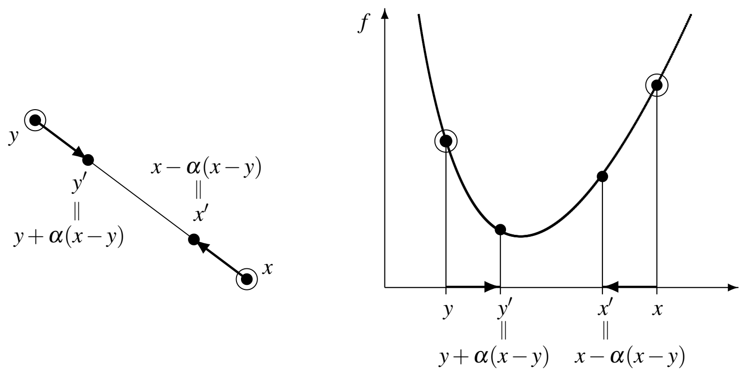


Fig. 1.13 Equidistance convexity

For a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ in discrete variables we simulate equidistance convexity (1.37) by moving a pair of points (x,y) to another pair (x',y') along the coordinate axes rather than on the connecting line segment. To be more specific, we consider two kinds of possibilities

$$(x',y') = (x - \mathbf{1}^i, y + \mathbf{1}^i) \quad \text{or} \quad (x',y') = (x - \mathbf{1}^i + \mathbf{1}^j, y + \mathbf{1}^i - \mathbf{1}^j) \tag{1.38}$$

with indices i and j such that $x_i > y_i$ and $x_j < y_j$; see Fig. 11.2. For a vector $z \in \mathbb{R}^n$ in general, define the *positive* and *negative supports* of z as

$$\mathrm{supp}^+(z) = \{i \mid z_i > 0\}, \qquad \mathrm{supp}^-(z) = \{j \mid z_j < 0\}. \tag{1.39}$$

Then (1.38) can be rewritten compactly as $(x',y') = (x - \mathbf{1}^i + \mathbf{1}^j, y + \mathbf{1}^i - \mathbf{1}^j)$ with $i \in \mathrm{supp}^+(x-y)$ and $j \in \mathrm{supp}^-(x-y) \cup \{0\}$, where $\mathbf{1}^0 = \mathbf{0}$.

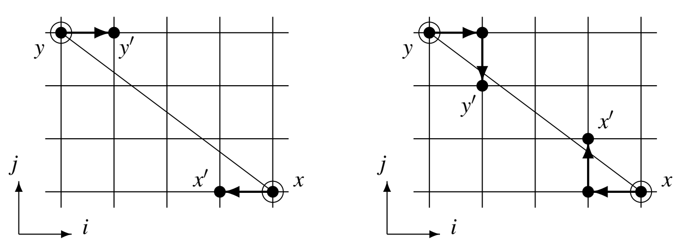


Fig. 1.14 Nearer pairs (x',y') in the definition of M$^\natural$-convex functions

As a discrete analogue of equidistance convexity (1.37) we consider the following condition: For any $x, y \in \mathrm{dom}_{\mathbb{Z}} f$ and any $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \mathbf{1}^i + \mathbf{1}^j) + f(y + \mathbf{1}^i - \mathbf{1}^j), \tag{1.40}$$

which is referred to as the *exchange property*. A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ having this exchange property is called *M$^\natural$-convex*. In the case of $n = 1$, M$^\natural$-convexity is equivalent to the condition (1.3).

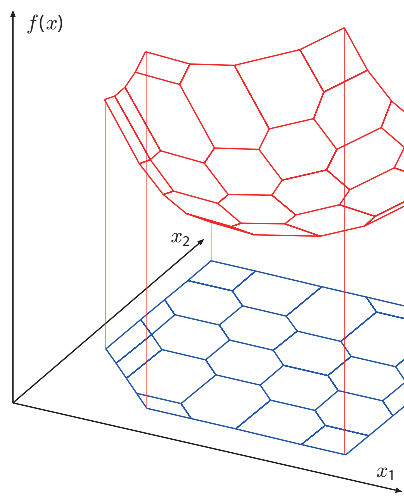


Fig. 1.15 An M$^\natural$-convex function ($n = 2$)

With this definition we can obtain the five desired properties as follows. Note that the conjugacy statement refers to L$^\natural$-convexity introduced in Section 1.3.4.

1. An M$^\natural$-convex function is convex-extensible.
2. For an M$^\natural$-convex function f, a point $x \in \mathrm{dom}_{\mathbb{Z}} f$ is a global minimizer of f if and only if it is a local minimizer in the sense that
$$f(x) \leq f(x - \mathbf{1}^i + \mathbf{1}^j) \qquad (\forall i, j \in \{0, 1, \ldots, n\}). \tag{1.41}$$
3. The integral conjugate $f^\bullet$ in (1.20) of an integer-valued M$^\natural$-convex function f is an integer-valued L$^\natural$-convex function. Furthermore, we have integral biconjugacy $f^{\bullet\bullet} = f$ under the fully-integral Legendre–Fenchel transformation (1.20).
4. A discrete separation theorem of the form of Theorem 1.7 holds for M$^\natural$-convex functions.
5. A Fenchel-type min-max duality of the form of Theorem 1.8 holds for integer-valued M$^\natural$-convex functions.

A function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is called *M-convex* if we can always choose $j \in \mathrm{supp}^-(x-y)$ (i.e., $j \neq 0$) in the exchange property (1.40). In other words, f is an M-convex function if and only if it is M$^\natural$-convex and $\mathrm{dom}_{\mathbb{Z}} f \subseteq \{x \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i = r\}$ for some $r \in \mathbb{Z}$. M-convex functions and M$^\natural$-convex functions are essentially equivalent in the sense that M$^\natural$-convex functions in n variables can be obtained as projections of M-convex functions in $n+1$ variables.

M-convex functions also enjoy the five key properties in slightly modified forms. The local minimality condition (1.41) should be changed to

$$f(x) \leq f(x - \mathbf{1}^i + \mathbf{1}^j) \qquad (\forall i, j \in \{1, 2, \ldots, n\}) \tag{1.42}$$

and the conjugate of an M-convex function is an L-convex function.

1.3.6 Discrete convex sets

The concepts of discrete convex sets are naturally induced from those of discrete convex functions as follows.

In the continuous case, the convexity of a set $S \subseteq \mathbb{R}^n$ can be characterized using its *indicator function* $\delta_S : \mathbb{R}^n \to \{0, +\infty\}$, which is defined as

$$\delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \notin S). \end{cases} \tag{1.43}$$

That is, a set S is convex if and only if its indicator function δ_S is convex.

For a set $S \subseteq \mathbb{Z}^n$ we may regard its indicator function as a function $\delta_S : \mathbb{Z}^n \to \{0, +\infty\}$ on $\mathbb{Z}^n$. Then the concepts of *L$^\natural$-convex sets* and *M$^\natural$-convex sets* can be defined as

$$S \text{ is an L}^\natural\text{-convex set} \iff \delta_S \text{ is an L}^\natural\text{-convex function}, \tag{1.44}$$

$$S \text{ is an M}^\natural\text{-convex set} \iff \delta_S \text{ is an M}^\natural\text{-convex function}. \tag{1.45}$$

Similarly we can define the concepts of *L-convex sets*, *M-convex sets*, and *integrally convex sets*. An L$^\natural$-convex (M$^\natural$-convex, L-convex, M-convex, or integrally convex) set S has the property

$$S = \overline{S} \cap \mathbb{Z}^n, \tag{1.46}$$

where $\overline{S}$ denotes the convex hull of S. This is sometimes referred to as the *hole-free property*.

For an L$^\natural$-convex function f, the effective domain $\mathrm{dom}_{\mathbb{Z}} f$ and the set of minimizers $\mathrm{argmin}_{\mathbb{Z}} f$ are L$^\natural$-convex sets. This statement remains true when L$^\natural$-convexity is replaced by M$^\natural$-convexity, L-convexity, M-convexity, or integral convexity.

Discrete convex sets are discussed fully in Chapter 10, and the discrete convexity of $\mathrm{dom}_{\mathbb{Z}} f$ and $\mathrm{argmin}_{\mathbb{Z}} f$ are treated in Chapters 11 and 13, respectively.

1.4 Comparison of Discrete Convex Functions

1.4.1 Inclusion relation among function classes

We have defined L$^\natural$-convex functions and M$^\natural$-convex functions by discretization of midpoint convexity (1.29) and equidistance convexity (1.37), respectively, to discrete midpoint convexity (1.30) and exchange property (1.40). This scheme is summarized in Fig. 1.16.

⟨Continuous Variables⟩ $f : \mathbb{R}^n \to \overline{\mathbb{R}}$		⟨Discrete Variables⟩ $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$
midpoint convex (1.29)	$\longrightarrow$	discrete midpoint convex (1.30)
$\Updownarrow$		(L$^\natural$-convex)
(ordinary) convex (1.1)		
$\Updownarrow$		(M$^\natural$-convex)
equidistance convex (1.37)	$\longrightarrow$	exchange property (1.40)

Fig. 1.16 Definitions of L$^\natural$-convexity and M$^\natural$-convexity by discretization

Figure 1.17 shows the inclusion relations among classes of discrete convex functions we have introduced. Integrally convex functions contain both L$^\natural$-convex functions and M$^\natural$-convex functions. L$^\natural$-convex functions contain L-convex functions as a special case. The same is true for M$^\natural$-convex and M-convex functions. The classes of L-convex functions and M-convex functions are disjoint, whereas the intersection of the classes of L$^\natural$-convex functions and M$^\natural$-convex functions is exactly the class of separable convex functions.

A function $f : 2^N \to \overline{\mathbb{R}}$ that assigns a real number (or $+\infty$) to each subset of $N = \{1,2,\ldots,n\}$ is called a *set function*, where notation 2^N means the set of all subsets of N, and hence $X \in 2^N$ is equivalent to saying that X is a subset of N. A set function f is said to be *submodular* if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (\forall X, Y \subseteq N), \tag{1.47}$$

where it is understood that the inequality is satisfied if $f(X)$ or $f(Y)$ is equal to $+\infty$.

A set function $f : 2^N \to \overline{\mathbb{R}}$ can be identified with a function $g : \mathbb{Z}^n \to \overline{\mathbb{R}}$ with $\mathrm{dom}\, g \subseteq \{0,1\}^n$ through the correspondence $f(X) = g(\mathbf{1}^X)$ for $X \subseteq N$. With this correspondence in mind we can say that submodular set functions are exactly L$^\natural$-convex functions on $\{0,1\}^n$, and *valuated matroids* (see Section 4.1 for definition) are exactly M-concave functions on $\{0,1\}^n$. Part II, consisting of Chapters 3 to 9, is devoted to the study of such set functions with discrete convexity/concavity.

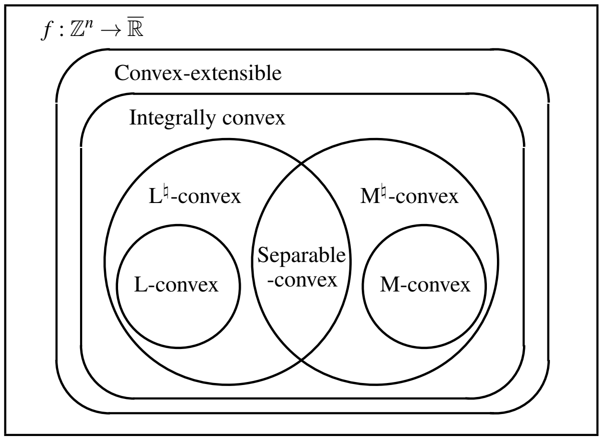


Fig. 1.17 Classes of discrete convex functions ($\mathrm{L}^{\natural}$-convex $\cap$ $\mathrm{M}^{\natural}$-convex $=$ separable convex)

Other kinds of discrete convex functions are also treated in this book, including multimodular functions (Section 11.6), globally and locally discrete midpoint convex functions (Section 11.7), M-convex functions on jump systems (Section 31.3), and L-convex functions on trees and graphs (Section 32.1). Multimodularity can be regarded as a variant of $\mathrm{L}^{\natural}$-convexity, since a function $f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is *multimodular* if and only if it can be represented as $f(x) = g(x_1,\ x_1 + x_2,\ \ldots,\ x_1 + \cdots + x_n)$ for some $\mathrm{L}^{\natural}$-convex function g. *Globally* (resp., *locally*) *discrete midpoint convex functions* are defined by weakening the condition for $\mathrm{L}^{\natural}$-convexity, that is, by imposing the discrete midpoint convex inequality (1.30) only when $\|x - y\|_\infty \geq 2$ (resp., $\|x - y\|_\infty = 2$). We often use "DMC" to mean "discrete midpoint convex(ity)."

Table 1.1 Various kinds of discrete convex functions

Convexity	Domain	Defining condition (roughly)
submodular set fn	2^N	$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$
valuated matroid	2^N	$f(X) + f(Y) \leq \max\{f(X - i + j) + f(Y + i - j)\}$
separable convex	$\mathbb{Z}^n$	$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$ (φ_i: convex)
integrally convex	$\mathbb{Z}^n$	Local convex extension is (globally) convex
$\mathrm{L}^{\natural}$-convex	$\mathbb{Z}^n$	$f(x) + f(y) \ \geq\ f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$
L-convex	$\mathbb{Z}^n$	$\mathrm{L}^{\natural}$-convex & linear in direction $\mathbf{1}$
$\mathrm{M}^{\natural}$-convex	$\mathbb{Z}^n$	$f(x) + f(y) \geq \min\{f(x - \mathbf{1}^i + \mathbf{1}^j) + f(y + \mathbf{1}^i - \mathbf{1}^j)\}$
M-convex	$\mathbb{Z}^n$	$\mathrm{M}^{\natural}$-convex & constant component-sum on $\mathrm{dom}\, f$
multimodular	$\mathbb{Z}^n$	$f(x) = g(x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_n)$, g: $\mathrm{L}^{\natural}$-convex
globally DMC	$\mathbb{Z}^n$	$f(x) + f(y) \ \geq\ f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$ ($\|x - y\|_\infty \geq 2$)
locally DMC	$\mathbb{Z}^n$	$f(x) + f(y) \ \geq\ f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$ ($\|x - y\|_\infty = 2$)
M-convex (jump)	$\mathbb{Z}^n$	$f(x) + f(y) \geq \min\{f(x \pm \mathbf{1}^i \pm \mathbf{1}^j) + f(y \mp \mathbf{1}^i \mp \mathbf{1}^j)\}$
L-convex	tree/graph	$f(x) + f(y) \ \geq\ f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$

* Valuated matroid is discrete concave

The discrete convex functions considered in this book are listed in Table 1.1 with a brief description of definitions. In addition to Fig. 1.17 we have the following inclusion relations among the function classes:

$$\{\text{separable convex}\} \subsetneqq \{\text{multimodular}\} \subsetneqq \{\text{integrally convex}\},$$
$$\{\text{L}^\natural\text{-convex}\} \subsetneqq \{\text{globally DMC}\} \subsetneqq \{\text{locally DMC}\} \subsetneqq \{\text{integrally convex}\},$$
$$\{\text{M}^\natural\text{-convex}\} \subsetneqq \{\text{M-convex (jump)}\} \not\subseteq \{\text{convex-extensible}\}.$$

The inclusion relations, including those in Fig. 1.17, will be proved in Chapter 11.

1.4.2 Comparion with respect to the key properties

We compare discrete convex functions with respect to the five crucial properties highlighted in this chapter. In Table 1.2 below, "Y" means "Yes, this function class has this property" and "***N***" means "No, this function class does not have this property." In the columns of "Biconjugacy" and "Fenchel duality," functions are assumed to be integer-valued and the fully-integral Legendre–Fenchel transformation (1.20) with $p \in \mathbb{Z}^n$ is used.[7]

Table 1.2 Five key properties of discrete convex functions

Discrete convexity	Convex-extension	Local/global minimality	Integral biconjugacy	Separation theorem	Fenchel duality
separable convex	Y	Y	Y	Y	Y
integrally convex	Y	Y	Y	***N***	***N***
$\text{L}^\natural$-convex	Y	Y	Y	Y	Y
L-convex	Y	Y	Y	Y	Y
$\text{M}^\natural$-convex	Y	Y	Y	Y	Y
M-convex	Y	Y	Y	Y	Y
multimodular	Y	Y	Y	Y	Y
globally DMC	Y	Y	Y	***N***	***N***
locally DMC	Y	Y	Y	***N***	***N***
M-convex (jump)	***N***	Y	***N***	***N***	***N***

[7] We also discuss conjugacy and Fenchel-type duality for real-valued functions in Chapters 16 and 18.

1.4.3 Operations on discrete convex functions

We compare the discrete convex functions with respect to admissible operations such as an origin shift of variables, a sign inversion of variables, scaling of variables and function values, addition and convolution of functions. While these operations are discussed in detail in Chapters 14 and 22, Tables 1.3 and 1.4 show a quick comparison. In these tables, "Y" means "Yes, this function class is closed under this operation" and "***N***" means "No, this function class is not closed under this operation."

Table 1.3 covers the following operations.

- An origin shift means $f(x+b)$ with an integer vector b.
- Two types of sign inversion of variables are distinguished. A simultaneous sign inversion means $f(-x_1,-x_2,\ldots,-x_n)$, and an independent sign inversion means $f(\pm x_1,\pm x_2,\ldots,\pm x_n)$ with an arbitrary choice of "+" and "−" for each variable.
- Permutation of variables means $f(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)})$ with a permutation σ of $(1,2,\ldots,n)$.
- Scaling of variables means $f(sx_1,sx_2,\ldots,sx_n)$ with a positive integer s. Note that the same scaling factor s is used for all coordinates.

Table 1.3 Operations on discrete convex functions via coordinate changes

Discrete convexity	Origin shift $f(x+b)$	Sign inversion simult. $f(-x)$	Sign inversion indep. $f(\pm x_i)$	Permutation $f(x_{\sigma(i)})$	Scaling $f(sx)$
separable convex	Y	Y	Y	Y	Y
integrally convex	Y	Y	Y	Y	***N***
L$^\natural$-convex	Y	Y	***N***	Y	Y
L-convex	Y	Y	***N***	Y	Y
M$^\natural$-convex	Y	Y	***N***	Y	***N***
M-convex	Y	Y	***N***	Y	***N***
multimodular	Y	Y	***N***	***N***	Y
globally DMC	Y	Y	***N***	Y	Y
locally DMC	Y	Y	***N***	Y	Y
M-convex (jump)	Y	Y	Y	Y	***N***

Table 1.4 covers the following operations.

- Value scaling means $af(x)$ with a nonnegative factor $a \geq 0$.
- Restriction and projection are defined with reference to a partition of the components of x into two parts[8] as $x=(y,z)$. We call $f(y,\mathbf{0})$ the *restriction* (or *section*) of f, and $\inf_z f(y,z)$ the *projection* (or *partial minimization*) of f.

[8] More precisely, "$x=(y,z)$" is a short-hand notation to mean that $x_i=y_i$ for $i\in Y$ and $x_i=z_i$ for $i\in Z$ for some partition of $\{1,2,\ldots,n\}$ into two disjoint nonempty subsets Y and Z. For $x=(x_1,x_2,x_3,x_4)$, for example, we can take $y=(x_2,x_3)$ and $z=(x_1,x_4)$.

- Addtion of two functions is defined, in an obvious way, by the addition of function values for each x. Two types of additions are distinguished: one is the addition of f with a separable convex function φ, denoted $f+\varphi$, and the other is the sum f_1+f_2 of two functions f_1 and f_2 in the same class. Since a linear function is separable convex, $f(x)+\langle p,x\rangle$ is a special case of $f+\varphi$.
- Convolution of two functions f and g is defined as $(f \Box g)(x)=\inf\{f(y)+g(z) \mid x=y+z,\ y,z\in\mathbb{Z}^n\}$. Two types of convolutions are distinguished: one is the convolution of f with a separable convex function φ, denoted $f \Box \varphi$, and the other is the convolution $f_1 \Box f_2$ of two functions f_1 and f_2 in the same class.

Table 1.4 Operations on discrete convex functions related to function values

Discrete convexity	Value scaling	Restriction	Projection	Addition		Convolution	
				separable	general	separable	general
	$af(x)$	$f(y,\mathbf{0})$	$\min_z f(y,z)$	$f+\varphi$	f_1+f_2	$f \Box \varphi$	$f_1 \Box f_2$
separable convex	Y	Y	Y	Y	Y	Y	Y
integrally convex	Y	Y	Y	Y	***N***	Y	***N***
L$^\natural$-convex	Y	Y	Y	Y	Y	Y	***N***
L-convex	Y	***N***	Y	Y	Y	Y	***N***
M$^\natural$-convex	Y	Y	Y	Y	***N***	Y	Y
M-convex	Y	Y	***N***	Y	***N***	Y	Y
multimodular	Y	Y	***N***	Y	Y	***N***	***N***
globally DMC	Y	Y	Y	Y	Y	***N***	***N***
locally DMC	Y	Y	Y	Y	Y	***N***	***N***
M-convex (jump)	Y	Y	Y	Y	***N***	Y	Y

(φ: separable convex on a specified domain)

1.5 History of Discrete Convex Analysis

In this section we briefly describe the history of discrete convex functions closely related to L-convex and M-convex functions. There are, however, many other studies of discrete convex functions that have been done independently of L- and M-convex functions, such as Girlich–Kowaljow [19], Hemmecke–Köppe–Lee–Weismantel [22], Hochbaum [28], Hochbaum–Shanthikumar [29], Ibaraki–Katoh [30], Lee–Leyffer [38], Miller [40], Onn [63].

The origin of L-convex and M-convex functions can be traced back to 1935, when the concept of matroids was introduced by Whitney [73] and Nakasawa [62]. The equivalence between submodularity of rank functions and exchange property of independent sets is the germ of the conjugacy between L-convex and M-convex functions in discrete convex analysis. The subsequent history is outlined in Table 1.5.

Table 1.5 Development of discrete convex functions

1935	matroid	Whitney [73], Nakasawa [62]
1965	submodular set function	Edmonds [8]
1969	convex-cost network flow	Iri [31]
1982	gross substitutes condition	Kelso–Crawford [33]
1983	**Submodularity and Convexity**	
	discrete separation theorem	Frank [12]
	Fenchel-type duality	Fujishige [14]
	convex extension	Lovász [39]
1985	multimodular function	Hajek [21]
1990	valuated matroid	Dress–Wenzel [6, 7]
1990	integrally convex function	Favati–Tardella [10]
1996	M-convex function	Murota [43]
1998	**Discrete Convex Analysis**	Murota [45]
1998	L-convex function	Murota [45]
1999	$\mathrm{M}^{\natural}$-convex function	Murota–Shioura [56]
2000	$\mathrm{L}^{\natural}$-convex function	Fujishige–Murota [17]
2000	polyhedral L-/M-convex function	Murota–Shioura [57]
2003	quasi L-/M-convex function	Murota–Shioura [58]
2004	L-/M-convex function on $\mathbb{R}^n$	Murota–Shioura [60, 61]
2006	M-convex function on jump systems	Murota [52]
2011	submodular function on trees	Kolmogorov [34]
2015	L-convex function on graphs	Hirai [23, 24, 25, 26]

In the late 1960s, J. Edmonds found a duality theorem on the intersection problem for a pair of (poly)matroids. This theorem, Edmonds' intersection theorem, shows a min-max relation between the maximum of a common independent set and the minimum of a submodular set function derived from the rank functions. The famous article of Edmonds [8] convinced us of the fundamental role of submodularity in discrete optimization. Analogies of submodular functions to convex functions and to concave functions were discussed at the same time. The min-max relation supported the analogy to convex functions, whereas some other facts pointed to concave functions. No unanimous conclusion was reached for a long time.

The relationship between submodular functions and convex functions was made clear in the early 1980s. The fundamental relationship between submodular set functions and convex functions, due to Lovász [39], says that a set function is submodular if and only if the Lovász extension (a specific piecewise-linear extension) of that function is convex. Reformulations of Edmonds' intersection theorem into a separation theorem for a pair of submodular/supermodular functions by Frank [12] and a Fenchel-type min-max duality theorem by Fujishige [14] indicate similarity to convex functions. The discrete mathematical content of these theorems, which cannot be captured by the relationship of submodularity to convexity, lies in the integrality assertion for integer-valued submodular/supermodular functions. Further analogy to convex analysis such as subgradients was conceived by Fujishige [15]. These developments in the 1980s led us to the understanding that (i) submodularity should be compared to convexity, and (ii) the essence of the duality for a pair of submodular/supermodular functions lies in the discreteness (integrality) assertion in addition to the duality for convex/concave functions.

The Lovász extension of a submodular set function is a convex function, which is necessarily a positively homogeneous function satisfying $f(\lambda x) = \lambda f(x)$ for $\lambda \geq 0$. This means that the convexity arguments about submodularity in the 1980s focus on a restricted class of convex functions. The relationship of submodular set functions to convex functions is generalized to the full extent by the concept of L-convex functions in discrete convex analysis.

Addressing the issue of local vs global minimality for functions defined on integer lattice points, Favati and Tardella [10] came up with the concept of *integrally convex functions* in 1990. This concept successfully captures a fairly general class of functions on integer lattice points, for which a local minimality implies the global minimality. Moreover, the class of *submodular integrally convex functions* (i.e., integrally convex functions that are submodular on integer lattice points) was investigated as a well-behaved subclass of integrally convex functions. It turned out later [17] that this concept is equivalent to $\mathrm{L}^\natural$-convex functions in discrete convex analysis.

We have so far seen major milestones towards L-convex functions, and are now turning to M-convex functions.

A weighted version of the matroid intersection problem was introduced by Edmonds [8]. The problem is to find a maximum weight common independent set (or a common base) with respect to a given weight vector. Efficient algorithms for this problem were developed in the 1970s by Edmonds [9], Lawler [37], Tomizawa–Iri [70], and Iri–Tomizawa [32] on the basis of a nice optimality criterion in terms of dual variables. The optimality criterion of Frank [11] in terms of weight splitting can be thought of as a version of such optimality criterion using dual variables. The weighted matroid intersection problem was generalized to the polymatroid intersection problem as well as to the submodular flow problem. It should be noted, however, that in all of these generalizations the weighting remained linear or separable convex.

The concept of *valuated matroids*, introduced by Dress and Wenzel [6,7] in 1990, provides a nice framework of nonlinear optimization on matroids. A valuation of a matroid is a nonlinear and nonseparable function on bases satisfying a certain exchange axiom. It was shown by Dress and Wenzel that a version of greedy algorithm works for maximizing a matroid valuation, and this property in turn characterizes matroid valuations.

Not only the greedy algorithm but the intersection problem extends to valuated matroids. The valuated matroid intersection problem, introduced by Murota [41], is to maximize the sum of two valuations. This generalizes the weighted matroid intersection problem since linear weighting is a special case of matroid valuation. Optimality criteria such as weight splitting as well as algorithms for the weighted matroid intersection are generalized to the valuated matroid intersection (Murota [42]). Analogy of matroid valuations to concave functions resulted in a Fenchel-type min-max duality theorem for matroid valuations (Murota [44]). This Fenchel-type duality is not a generalization nor a special case of Fujishige's Fenchel-type duality for submodular functions, but these two are generalized and unified into a single min-max equation, which is the Fenchel-type duality theorem in discrete convex analysis.

The analogy of valuated matroids to concave functions led to the concept of *M-convex/concave functions* in Murota [43], 1996. M-convexity is a concept of "convexity" for functions defined on integer lattice points in terms of an exchange axiom, and affords a common generalization of valuated matroids and (integral) polymatroids. A valuated matroid can be identified with an M-concave function defined on $\{0,1\}$-vectors, and the base polyhedron of an integral polymatroid is a synonym for a $\{0,+\infty\}$-valued M-convex function. The valuated matroid intersection problem and the polymatroid intersection problem are unified into the M-convex intersection problem. The Fenchel-type duality theorem for matroid valuations is generalized for M-convex functions, and the submodular flow problem to the M-convex submodular flow problem (Murota [47]), which involves an M-convex function as a nonlinear cost. The nice optimality criterion using dual variables survives in this generalization. Thus, M-convex functions yield fruitful generalizations of many important optimization problems on matroids.

The two independent lines of developments, namely, the convexity argument for submodular functions in the early 1980s and that for valuated matroids and M-convex functions in the early nineties, were merged into a unified framework of "Discrete Convex Analysis," advocated by Murota [45] in 1998. The concept of *L-convex functions* was introduced as a generalization of submodular set functions. L-convex functions form a conjugate class of M-convex functions with respect to the Legendre–Fenchel transformation. This completes the picture of conjugacy advanced by Whitney [73] as the equivalence between submodularity of the rank function of a matroid and exchange property of independent sets of a matroid. The duality theorems generalize to L-convex and M-convex functions. In particular, the separation theorem for L-convex functions is a generalization of Frank's separation theorem for submodular set functions.

Ramifications of the concepts of L- and M-convexity followed. *M*$^\natural$*-convex functions*, introduced by Murota–Shioura [56], are essentially equivalent to M-convex functions but are often more convenient in applications. *L*$^\natural$*-convex functions*, due to Fujishige–Murota [17], are an equivalent variant of L-convex functions. It turned out [17] that L$^\natural$-convex functions are exactly the same as submodular integrally convex functions that had been introduced by Favati–Tardella [10] in 1990. Quasi L-convex and M-convex functions are introduced in Murota–Shioura [58], and M-convex functions on jump systems in Murota [52].

While the functions described above are defined for discrete variables belonging to $\mathbb{Z}^n$ or $\{0,1\}^n$, it is also possible to define L- and M-convexity for functions in real or continuous variables belonging to $\mathbb{R}^n$. This amounts to investigating convex functions with some additional combinatorial structures. L- and M-convexity are defined for polyhedral functions (Murota–Shioura [57]), quadratic functions (Murota–Shioura [59]), and closed convex functions (Murota–Shioura [60, 61]). They form subclasses of ordinary convex functions, and the conjugacy under the Legendre–Fenchel transformation holds between L-convex and M-convex functions.

Since 2010, discrete convex functions on graph structures have been investigated. The concept of submodular function on trees is formulated by Kolmogorov [34] as a framework unifying L$^\natural$-convex functions and bisubmodular functions. A theory of L-convex functions on graphs has been developed by Hirai [23,24,25,26], motivated by combinatorial dualities in multi-commodity flow problems and complexity classification of facility location problems on graphs. This theory exhibits substantial progress of discrete convex analysis, which may be called "discrete convex analysis beyond $\mathbb{Z}^n$."

We have outlined the development of the framework of discrete convex functions starting from matroids and submodular set functions. These studies are mostly driven by mathematical interest or aesthetics. However, substantial results on discrete convex functions have also been obtained in the literature of more practical disciplines.

In the late 1960s, Iri [31] made a comprehensive study of nonlinear electric networks, where the role and significance of convexity are considered in combination with discrete structures stemming from the underlying graphs. This study, with a dual view of mathematics and physics, offers a major guiding principle to discrete convex analysis.

Discrete convex functions appear naturally in operations research. In queueing theory, Hajek [21] introduced the concept of *multimodular functions* in 1985. Multimodular functions are further investigated in the literature of discrete event systems (Altman–Gaujal–Hordijk [1], Glasserman–Yao [20]). It turned out later [51] that this concept is equivalent to L$^\natural$-convexity through a simple transformation of variables. L$^\natural$-convex functions are used and studied also in inventory theory (Simchi-Levi–Chen–Bramel [68]).

In economics and game theory, Kelso and Crawford [33] introduced *gross substitutes condition* in 1982. Subsequent studies in the literature of mathematical economics revealed that this condition is crucial for the existence of economic or game-theoretic equilibria. In 2003, Fujishige and Yang [18] pointed out that this condition is equivalent to $M^\natural$-concavity, which triggered active interaction between mathematical economics and discrete convex analysis (Murota [55], Shioura–Tamura [67], Tamura [69]).

It is emphasized that the theory of discrete convex analysis have benefited a lot from interactions with application fields outside optimization. Some concepts, theorems and algorithms were discovered in the process of solving concrete problems what had denied solution by the then existing tools. In particular, the valuated matroid intersection problem [41, 42] was formulated and solved in an effort of solving a certain problem in matrix theory.[9]

Finally, we mention that the earlier development of discrete convex analysis is presented in monographs [49] and [50] published in 2001 and 2003, and in [16, Chapter VII] published in 2005. More recent surveys are given in [54, 55].

[9] More specifically, the problem is to design an efficient algorithm to compute the degree of determinants of a mixed polynomial matrix [46], which is described also in [48, Chapter 6] and [50, Chapter 12].

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