RIMS Summer School (COSS 2018), Kyoto, July 2018
Problems for "Discrete Convex Analysis" (by Kazuo Murota)

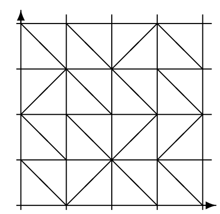
Solve the problems marked by (COSS) .

Problem 1. (COSS) Prove that a function $f : \mathbf{Z}^2 \to \mathbf{R}$ defined by $f(x_1, x_2) = \varphi(x_1 - x_2)$ is an L$^\natural$-convex function, where $\varphi : \mathbf{Z} \to \mathbf{R}$ is a univariate discrete convex function (i.e., $\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)$ for all $t \in \mathbf{Z}$).

Problem 2. Prove that a function $f : \mathbf{Z}^2 \to \mathbf{R}$ defined by $f(x_1, x_2) = \varphi(x_1 + x_2)$ is an M$^\natural$-convex function, where $\varphi : \mathbf{Z} \to \mathbf{R}$ is a univariate discrete convex function.

Problem 3. (1) Show that a function $f(x_1, x_2)$ is M$^\natural$-convex if and only if $f(x_1, -x_2)$ is L$^\natural$-convex.
(2) Is there any such correspondence for functions in three or more variables?

Problem 4. (COSS) Find an integrally convex function that corresponds to the triangulation to the right:



Problem 5. (COSS) Prove that $f(x) = \max\{x_1,\ x_2, \ldots,\ x_n\}$ is an L-convex function.

Problem 6. (COSS) Let $G = (V, E)$ be a connected graph. Define a set function f by: $f(X) = 0$ if X is the edge set of a spanning tree, and $f(X) = +\infty$ otherwise. We may regard f as $f : \mathbf{Z}^E \to \mathbf{Z} \cup \{+\infty\}$. Show that f is an M-convex function.

For a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ and a family of univariate discrete convex functions $\varphi_A : \mathbf{Z} \to \mathbf{R}$ indexed by $A \in \mathcal{F}$, we consider a function defined by

$$f(x) = \sum_{A \in \mathcal{F}} \varphi_A(x(A)) \qquad (x \in \mathbf{Z}^n), \tag{1}$$

where $x(A) = \sum_{i \in A} x_i$. A function $f : \mathbf{Z}^n \to \mathbf{R}$ is called laminar convex if it can be represented in this form for some laminar family $\mathcal{F}$ and φ_A $(A \in \mathcal{F})$.

Problem 7. Prove that a laminar convex function is M$^\natural$-convex.

In Problems 8–11, we consider a quadratic function in three variables $f(x) = x^\top A x$ $(x \in \mathbf{Z}^3)$ defined by a 3×3 symmetric matrix $A = (a_{ij})$.

Problem 8. (1) Find a necessary and sufficient condition on (a_{ij}) for $f(x)$ to be submodular.
(2) When $f(x)$ is submodular, is the matrix A positive semidefinite?

Problem 9. (1) Find a necessary and sufficient condition on (a_{ij}) for $f(x)$ to be L$^\natural$-convex.
(2) When $f(x)$ is L$^\natural$-convex, is the matrix A positive semidefinite?

Problem 10. (1) Show that $f(x)$ is an M$^\natural$-convex function if and only if (i) $a_{ii} \geq a_{ij} \geq 0$ for all (i, j), and (ii) the minimum among the three off-diagonal elements, a_{12}, a_{23}, a_{13}, is attained by at least two elements.
(2) When $f(x)$ is M$^\natural$-convex, is the matrix A positive semidefinite?

Problem 11. (1) Is $f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_1 + x_3)^2$ laminar convex?
(2) Is this function M$^\natural$-convex?
(3) Prove that a quadratic function $f(x)$ ($x \in \mathbf{Z}^3$) is M$^\natural$-convex if and only if it is laminar convex[1].

Problem 12. (1) Show that $f(x_1, x_2, x_3) = a(x_1 + x_2)^2 + b(x_2 + x_3)^2 + c(x_1 + x_3)^2$ with randomly chosen $a, b, c > 0$ is not an M$^\natural$-convex function.
(2) Show that, under some "nondegeneracy assumption," a function $f(x)$ of the form (1) is M$^\natural$-convex only if $\mathcal{F}$ is a laminar family.

Problem 13. Miller's paper (1971) in inventory theory dealt with the function:

$$f(x) = \sum_{k=0}^{\infty} \left(1 - \prod_{i=1}^{n} F_i(x_i + k) \right) + \sum_{i=1}^{n} c_i x_i \qquad (x = (x_1, \ldots, x_n) \in \mathbf{Z}^n_+), \tag{2}$$

where $F_1, \ldots, F_n$ are cumulative distribution functions of Poisson distributions (with different means), and $c_1, \ldots, c_n$ are nonnegative real numbers. Prove that this function is L$^\natural$-convex.

For a matroid on V, the rank function ρ is defined by

$$\rho(X) = \max\{|I| \mid I \text{ is an independent set, } I \subseteq X\} \qquad (X \subseteq V). \tag{3}$$

Problem 14. (COSS) Let ρ be a matroid rank function on V, and define $f : \{0, 1\}^V \to \mathbf{Z}$ by $f(\mathbf{1}^X) = \rho(X)$ for $X \subseteq V$, where $\mathbf{1}^X$ denotes the characteristic vector of X.
(1) Prove that f is L$^\natural$-convex.
(2) Prove that f is M$^\natural$-concave.
(3) Prove that $f^\bullet(\mathbf{1}^X) = |X| - f(\mathbf{1}^X)$ for $X \subseteq V$, where $f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \{0, 1\}^V\}$.

Problem 15. Let ρ_1 and ρ_2 be the rank functions of two matroids on V. For the rank of the union matroid, the following formula is known:

$$\max_X\{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y\{\rho_1(Y) + \rho_2(Y) - |Y|\} + |V|. \tag{4}$$

Relate this formula to the Fenchel min-max duality in discrete convex analysis.

Problem 16. (COSS) Let $f : \mathbf{Z}^3 \to \mathbf{Z} \cup \{+\infty\}$ be defined by $f(0, 0, 0) = 0$ and $f(1, 1, 0) = f(0, 1, 1) = f(1, 0, 1) = 1$, with $\mathrm{dom}\, f = \{(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)\}$.
(1) Show that f is integrally convex.
(2) Show that the subdifferential of f at $x = \mathbf{0}$ is given as

$$\partial f(\mathbf{0}) = \{p = (p_1, p_2, p_3) \in \mathbf{R}^3 \mid p_1 + p_2 \leq 1, p_2 + p_3 \leq 1, p_1 + p_3 \leq 1\}.$$

(3) Show that $\partial f(\mathbf{0})$ is not an integer polyhedron.
(4) Show that $\partial f(\mathbf{0})$ contains an integer point.

Problem 17. (COSS) Let $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ be an integer-valued integrally convex function with $f(\mathbf{0}) < +\infty$.
(1) Show that $\partial f(\mathbf{0})$ is nonempty.

[1]This statement is true for general n. That is, a quadratic function in n integer variables is M$^\natural$-convex if and only if it is laminar convex.

(2) Show that $\partial f(\mathbf{0})$ is given as $\partial f(\mathbf{0}) = \{p \in \mathbf{R}^n \mid \sum_{j=1}^{n} y_j p_j \leq f(y) - f(\mathbf{0}) \ \ (\forall y \in \{-1, 0, +1\}^n)\}$.

(3) Suppose that we apply the Fourier–Motzkin elimination to the system of inequalities $\sum_{j=1}^{n} y_j p_j \leq f(y) - f(\mathbf{0})$ indexed by $y \in \{-1, 0, +1\}^n$. Show that we do not need to generate new inequalities in the elimination process.

(4) Show that $\partial f(\mathbf{0})$ contains an integer vector.

Problem 18 (Research Problem (COSS))**.** The integral biconjugacy for integrally convex functions implies that there is a one-to-one correspondence between the class $\mathcal{F}_{\mathrm{IC}}$ of integer-valued integrally convex functions and the class of their integral conjugates $\mathcal{F}_{\mathrm{IC}}^{\bullet} = \{f^{\bullet} \mid f \in \mathcal{F}_{\mathrm{IC}}\}$. Give a direct characterization of $\mathcal{F}_{\mathrm{IC}}^{\bullet}$.

The steepest descent algorithm for an L$^\natural$-convex function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ reads as follows ($\mathbf{1}^X$ means the characteristic vector of a set $X \subseteq \{1, 2, \ldots, n\}$):

Step 0: Set $p := p^\circ$ (initial point).
Step 1: Find $\sigma \in \{+1, -1\}$ and X that minimize $g(p + \sigma \mathbf{1}^X)$.
Step 2: If $g(p + \sigma \mathbf{1}^X) = g(p)$, then output p and stop.
Step 3: Set $p := p + \sigma \mathbf{1}^X$ and go to Step 1.

In Problems 19 and 20 we consider the behavior of this algorithm when $n = 2$.

Problem 19. Define $g : \mathbf{Z}^2 \to \mathbf{R}$ by $g(p_1, p_2) = \max(0, -p_1 + 2, -p_2 + 1, -p_1 + p_2 - 1, p_1 - p_2 - 2)$.

(1) Verify that g is L$^\natural$-convex.

(2) Find the set, say, S of the minimizers of g. Draw a figure, indicating S on the lattice $\mathbf{Z}^2$.

(3) Take an initial point $p^\circ = (0, 0)$. Which minimizers are possibly found? Is the number of iterations constant, independent of the generated sequences of vector p? How is the number of iterations related to the ℓ_∞-distance from p° to S?

(4) Take another initial point $p^\circ = (1, 4)$. Which minimizers are possibly found? Is the number of iterations equal to the ℓ_∞-distance from p° to S?

Problem 20. Let $g : \mathbf{Z}^2 \to \mathbf{R}$ be an L$^\natural$-convex function that has a minimizer; denote by S the set of its minimizers. Give an expression for the number of iterations in terms of p° and S.

Problem 21 (M-minimizer cut theorem)**.** Let $f : \mathbf{Z}^n \to \mathbf{R}$ be an M-convex function such that $\arg\min f \neq \emptyset$. Take any $x \in \mathrm{dom}\, f$ and $i \in \{1, 2, \ldots, n\}$, and let $j \in \{1, 2, \ldots, n\}$ be such that $f(x - \mathbf{1}^i + \mathbf{1}^j) = \min_{1 \leq k \leq n} f(x - \mathbf{1}^i + \mathbf{1}^k)$. Prove that there exists $x^* \in \arg\min f$ such that $x_j^* \geq x_j + 1$ in the case of $i \neq j$ and $x_j^* \geq x_j$ in the case of $i = j$.

(END of Problems)